The Mathematics Teacher

OCTOBER 1956

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Time is running out[1]

RUDOLPH E. LANGER, *University of Wisconsin. Madison, Wisconsin.*

The importance of mathematics and mathematics teaching to our way of life is forcefully underscored. Mathematics IS *the Queen of the Sciences.*

AT A CONVENTION of teachers it is appropriate that much of the discussion should concern itself with the know-how of the teacher's job. Changes in course content, novel teaching materials, new methods of presentation, and matters of that sort are all of first-rate importance. They are concrete and specific. To keep oneself informed of them is the only means for keeping one's classroom practice alive and up to date.

I am not going to talk to you about such matters, however. If I did, you would certainly have every right to question my qualifications to do so. For, insofar as high school practices are concerned, I am an outsider. In this dilemma I have had to seek some other subject for this address. What I have decided is to speak to you of some matters that concern us all as members of the mathematics teaching profession, rather than as individuals. We, as a professional group, fill a certain social niche. Is the significance of this niche waxing or waning in the contemporary scene? Is our work gaining or losing in importance? What is the trend of our professional destiny? There is a trend to it, as there is to all human endeavor. To be a mathematics teacher is something different today than it was ten years ago. The expectation is that ten years hence it will be something different again.

I take it to be an axiom, or, if you will, a self-evident truth, that to be a really inspiring mathematics teacher one ought to be aware of what is going on in mathematics, and to have a pretty good appreciation of it. I am not thinking at this point of what is going on in mathematics in the sense of its own inner growth and development as a body of doctrine and theory. That might be a little too abstruse to discuss here. What I am thinking of, is what is happening with mathematics as a social force. It is important for us to be alert to that. It will qualify our responses when students want to know what mathematics is good for. It will help us in giving students sound guidance. It will affect our confidence in our subject, and if we do not have confidence, what, in the way of enthusiasm, could we bring to our classes.

In these general terms, all that I have said applies, of course, to any subject in the school curriculum. As it happens, however, it applies, in the immediate present, with very special emphasis to our subject. Because mathematics, in the contemporary scheme of things, is being instrumental in revolutionizing the whole patterns and practices of life throughout the world. We ourselves are subject to its effects. The mathematicians and physicists created the atomic bomb, but the bomb respects them no more than any others.

It is not many years ago that educational fadists discounted mathematics. With a few exceptions, they minimized its value for students. The time allotment given to mathematics in the curriculum was cut down. The favor went to other

[1] A paper delivered at the annual meeting held at Milwaukee, Wisconsin, April 11–14.

subjects that were regarded as more productive of good citizenship, and better designed to fit students to the environment in which they would have to earn a living. Mathematics teachers were on the defensive then. They knew such arguments were feeble. But the popular tide was running against them. Few of us foresaw then how quickly we would be vindicated, and how completely the asininity of the detractors of mathematics would be shown up. It hardly seems longer ago than yesterday that a change in tune was heard. Since then the crescendo has risen. Today it stands revealed, for even the most stupid to recognize, that mathematics has all the while been redirecting our whole social effort, and has been fundamentally revamping the environment in which so many unseeing, so-called "educational leaders" had preached its lesser importance.

In the course of history, mathematics has not often been a popular subject. It has never been easy, and the average human animal has never been anxious to exhaust itself in the difficult task of thinking. But as we look back upon history, it is easy to see that mathematics has always been a very powerful agent in directing the evolution of our cultural patterns. This is true today as it has never been true before. Because of that, we now hear on all sides of a shortage of mathematically trained personnel. And we also hear of the shortage of teachers to train that personnel, which seems now to be critically essential to our national progress, safety, and economy. This is what I would like to discuss with you.

The ability to invent and use mathematics, is the special genius of the human being. By it he is able to go beyond the limitations that nature has imposed upon him. By mathematics he extrapolates from the range of his actual experiences. Mathematical formulations of physical laws are the means for inferring future certainties out of past experiences. And they are also the means for immeasurably magnifying our faculties and perceptions, and for almost limitlessly refining them. They make it possible for us to deal with invisible things, and to anticipate nature in innumerable ways to our advantage.

Now the fact is that in the last half-century, the sciences, the physical sciences especially, have concerned themselves more and more with things that cannot be seen. No one sees the flow of an air stream or of a jet of gas, and no one sees an electromagnetic wave or a cosmic ray. No microscope can show us the structure of an atom. But by the use of mathematics the scientist has seen these things, and has seen them clearly enough to discover their laws. Mathematics has thus increased our knowledge. And the technologist, who always follows in the scientist's footsteps, has put this knowledge to use.

What this has done to our everyday environment, and to the tempo and tenor of our lives, is to be seen all about us. It has given us instruments and engines and gadgets in such variety and abundance, that our appreciations of the wonders of them are wholly blunted. We take the most marvelous events as matters of course, hardly thinking twice to see airplanes weighing tons upon tons taking their support from the insubstantial air. Nor are we astonished to hear music recreated in all its graceful rhythms and harmonies at places remote from its generation.

The invention of all these things has changed our scale of living, and in virtue of that has revolutionized our economy. We have a continuing hunger for new goods and contraptions. The production of these spawns new industries and magnifies and diversifies employment. Although you will, of course, know these things, you may yet be surprised at the tempo at which these developments are, even at this moment, proceeding. The use of electrical power in this country is 200 times as great now as it was 50 years ago. One large manufacturing corporation, the Johns-Manville Company, reports that half its

sales are in products that are new or greatly improved since 15 years ago. The Monsanto Chemical Company reports that 40 per cent of its profits come from products it did not even make 10 years ago.

Phenomenal as this rapid-fire change in the civilian industrial evolution is, it is even being outstripped by the military technology. The inventions of radar, jet propulsion engines, rocket motors, and atomic bombs were all the consequences of highly mathematical investigations. They are only the more conspicuous military innovations. Along with them the mathematical theories of elasticity, fluid dynamics, and combustion have effected immense improvements in the design of all kinds of military engines. The mathematical theories of statistics and of games meanwhile continue to revamp the organizational, the economic, and the tactical aspects of war and defense. The very latest engine of today will be obsolete next year, and the advance which leaves it behind is under the complete command of mathematics.

The scientific effort by which all this is being brought about is commensurate with the result. The goal of industry is profit. And where a company exists for profit, it can be depended upon to spend only where a greater return can be expected. How certain the pay-off of scientific research is recognized to be, is shown therefore by the willingness of industry to spend money for it. Forty years ago the number of research laboratories in this country was about 100. Today it is over 3000. The amount of money spent for research today is 23 times what it was in 1930. The Du Pont Company puts 61 million dollars a year into its research laboratory. The Bell Telephone Laboratory has a budget of 24 million dollars and a staff of 6000 people.

The sheer volume of mathematics that must be done increases with every year That is so, both at the upper level of science and at the lower level of business. At both levels we have passed the point at which human hands, with pencil and paper and moderate computing aids, can keep up with it. The last ten years have therefore brought forth the development of phenomenal machines for calculation, and for the arithmetical processing of business data. The recent development of the electronic computer is a typical example of what is going on in the technological world. Since this is what I have been speaking of, I shall just spend a few words on it. Inasmuch as these machines are symbolic of man's employment of mathematics to do mathematics for him, their development has the character of an ultimate conquest by man over his own natural limitations.

Computing machines in which mechanical elements perform the operations of arithmetic have long been available. Such machines are limited in speed, because mechanical parts, like levers and wheels, move with a certain slowness. This slowness is no great drawback in a desk calculator of the familiar kind, for in that each step is dictated to the machine by a human operator, and the machine performs each operation before the next one is dictated. Among the mathematical problems that science is now confronted with, however, there are many that require millions of multiplications and additions, etc., with very large numbers. For such problems a mechanical computer is inadequate, since it could produce the needed results only after months, or even years, of steady operation. In the electronic calculator wire circuits have replaced the mechanical parts, and the operating elements are minute electric pulses. The electric pulse, unlike the lever or wheel, is lightning fast. The machine deals with large numbers of 13 digits (remember, one million is only a 7-digit number). It multiplies together two 13-digit numbers in one 30,000th of a second, and it finds the sum of two such numbers in one 60,000th of a second.

Of course, no human operator can guide such a machine step by step. The machine is designed, therefore, to receive at one stroke the complete instructions for

doing a whole problem. It then functions as its own dictator in its step-by-step calculations. In the largest machines that are now commercially available, the volume of these instructions can run to the equivalent of 28,000 printed pages, and the machine carries these instructions out at about 14,000 times the rate at which an expert typist could type them out. In one second the machine does what it would take a skillful human calculator a week to do. And while it computes at this speed, the machine keeps account of the decimal point and of plus and minus signs. It checks itself at each step and carries out millions of operations without ever a mistake. To compute the path of a rifle bullet from the physical characteristics of the gun and the prevailing atmospheric conditions is a long and tedious step-by-step piece of work. The electronic computer, when set to this task, has it all completed before the bullet strikes its target.

An immediate effect of the accentuated role of technology in the life of a modern nation has been to multiply many times over the need for people with mathematical training. In industry, and in both the civilian and the military branches of the government, the present demand for mathematical skills is entirely disproportionate to what it was 10 years ago, and even more so to what it was 15 years ago. The time is not long past when college majors in mathematics were almost all absorbed by the teaching profession. Other opportunities for the use of their specialized knowledge were then very rare. Today the situation is as different as it possibly could be. Every college teacher is solicited long before graduation time for recommendations of students who are taking mathematics courses. In great numbers industrial firms send recruiting officers to the college campuses to offer employment to students of mathematics, science, and engineering. The salaries offered, and the working conditions, would have seemed incredible a few years ago. Yet the number of students that can be produced is far short of what is needed. There is no prospect, whatever, that the pace of technological development will slacken off. On the contrary, every indication is that it will increase. It would do so, even if it were not spurred on, as it presently is, by considerations of national survival.

Our country has enjoyed a unique advantage in the past. It has been a young country with a fresh outlook, among less enterprising and more tradition-bound competitors. We are no longer so favored. The Soviet Union confronts us with resources exceeding our own. It is governed by ideologies that are basically repugnant to us. That country also has the advantage. We cannot look upon it as merely a grown-up Russia of former times. It has divorced itself from its past and has taken on an outlook and enterprise that dates even more recently than ours. The threat which this holds forth to us is pointed up by the fact that this formidable rival has entered the lists on our own ground, namely in the field of science and technology. And it is doing this with a seriousness of purpose that calls for our deep apprehension. It menaces us with the loss of leadership in precisely the points on which our greatest reliance has been banked.

Good as it is, it appears that our technological prowess is not good enough. The stimulation of it is a matter to which we must give our very best thought. Without mathematics there is no technology and no modern physical science. Therefore, to do something about science, one must do something about mathematics. That is why the matter is one which merits our concern. The country is concerned about it. The National Research Council and the American Association for the Advancement of Science, as well as other foundations and societies, have engaged the best available people to give their attention to this matter. The President has just appointed a "National Committee for the Development of Scientists and Engineers." This was an action that is wholly without

precedent in our educational history. The public is still, in large part, uninformed of the facts and unaware of their significance. The newspaper and magazine presses are, however, now doing much to correct this. The schools would be wholly unjustified in ignoring or minimizing the situation.

The Soviet system of elementary and secondary education consists of a compulsory elementary program extending from the ages 7 to 14, and a secondary program extending to the age 17. During the elementary part, 35 per cent of the pupils' time is applied to the study of mathematics and science, and in the secondary part this is stepped up to 41 per cent. The curricula of the schools and the examinations given in them are uniform for the whole country. A high and rigid standard of student application can therefore be maintained. Students who do not proceed to the secondary school are required to attend technical schools, which prepare them for jobs as machinists and technicians of other kinds. Every student who finishes the secondary course has had classes in algebra, geometry, and trigonometry and has had four years of physics and four of chemistry. There are 33 Soviet universities and about 750 professional schools, most of these latter being schools of education, medicine, and engineering. Strong incentives are held out for students to proceed to higher education. Almost all students at that level are subsidized, and the larger subsidies go to the students with the better academic records. Jobs need not be looked for after graduation, because trained people are treated as valuable national assets. The salaries of scientists are relatively much higher than they are with us. A person of intellectual distinction therefore enjoys a high degree of public respect, and is afforded a life of considerable comfort.

We decry the regimentation upon which a system of training such as that of the Soviets is based. We regard it as preferable to hold to a more liberal ideal of education. Nevertheless, pitted as we are in competition with the product of the Soviet system, it would be unpardonably foolhardy for us to refuse to take stock of our own practice, and to recognize the faults which it probably has.

Where there is freedom there is likely to be wastefulness. And there seems to be a rapidly growing conviction that our school practices are altogether too wasteful—I mean of human talent. That is precisely the wastefulness that we can least afford. Apparently dependable reports show that the Soviet Union equaled our output of 30,000 engineers in 1952, whereas last year, in 1955, it outmatched our 23,000 engineer graduates by the immensely larger number of 63,000. It is known to be the objective of the Soviet leaders, not only to exploit all the technological possibilities of their own country to win and maintain a superiority over us, but to produce indoctrinated engineers in large numbers for export to underdeveloped countries elsewhere. If they do that, it is to be feared that the American dollar, however lavishly it may be given away, will come out only second best in the race.

It has been our tenet all along to extend the maximum educational opportunity to the whole of our population. This is a tenet that has, on the whole, served us well; one which we are certainly not now ready to abandon. Unfortunately, we have too often thought to live up to this tenet fully by merely keeping as many youngsters as possible in school for the maximum number of years. We have tended, in short, to interpret *maximum opportunity* in quantitative terms, rather than in qualitative. The effect of this has been to gear our school standards to a level of mediocrity. This sacrifices, to the large mass of the intellectually ungifted, that minority of gifted students upon which the future welfare of the whole community will depend. Students are required to work at their studies only as hard as it is feasible to make the unambitious ones work. For fear of leaving some stupid ones behind, we lose the interest of the superior ones

and allow them to be bored and to form wholly false notions of the responsibilities to which they should hold themselves up. The mission of the schools, namely to prepare pupils to assume their roles in life, has been frequently interpreted to mean preparation for life at the dead level where ideas never obtrude themselves, and existence conforms to an unimaginative and imitative routine

In the Soviet schools, they are not doing that. They have designed their practices to single out the capable students and to spur them on to make the most of their talents. They believe that in this way the whole nation is best served, and who shall say that they are wrong! We shall be wise, at all events, to remedy the faults we already recognize in our own procedures, and which presage for us a growing disadvantage.

In the first place, then, let us throw out the contention that mathematics courses offer something of value only to the college-bound student. There is, even today, a great field of usefulness for the mathematical skills which a high school training can give. This field will certainly be much wider in the future than it is today, and it is the future in which our pupils will live out their careers. Many of today's high school students will lead lives extending into the next century. Are we to believe, that, in these rapidly changing times, it is really profitable for them to spend their efforts upon subjects that are rich only in concrete and immediate practicalities of the present? Mathematics is an enduring subject, and it has at least the advantage that what is learned of it today will not be of merely historical interest a generation hence.

The day in which a moderate mathematical education had little commercial value, and advanced training was only a qualification for teaching, has gone by. Good mathematics talent, trained to any level, is in demand. There is simply no substitute for it. Machines can be made to do mathematics, but they will never minimize the need for mathematicians. For even the most imposing machine is not a "giant brain" as the newspapers have liked to call it. It is only an instrument, and if the mathematician omits a single instruction, even in the slightest detail, the machine is dumbfounded and completely helpless. The only thing it can do of itself is to stop.

In the second place, let us seek to curtail the diversion of talent from the high school that is now taking place. According to the Department of Health, Education and Welfare, about 60,000 high-quality students drop out of the high schools each year before graduation. At the same time 150,000 capable graduates do not go on to college. We must find out the prevailing reasons for these losses. These students will not be the ones who find the work too hard. Is it that many of them find school not worthwhile, because, in the absence of a challenge to their capabilities, their remaining in school seems futile? That would be a correctible thing, though it might not be easy to correct it. Inspiring teaching would be the means of correcting it, and that will not be forthcoming where teachers are loaded with too many and too large classes. Nor will it be forthcoming if teaching salaries are not brought up to levels where the profession is attractive to strong and capable personalities.

If the cause is rooted in financial need, it is again correctible. While it is true that scholarship provisions for college and further advanced study are much more abundant and ample than they used to be, it is nevertheless also true that these provisions should be extended much further on a national scale. Even so, however, losses from high school on this ground can hardly reflect very effective student guidance. For even now masses of young people, wholly without funds, succeed in carrying on their studies.

Let me, by way of conclusion, say something to you as mathematics teachers. Everything I have said should be significant to you. It should give you satisfaction and inspiration. We are living in a turbu-

lent age, in which nations are made and broken. Changes of social, economic, and material conditions have been greater in the past 40 years than they were in the preceding 400. And in this virtual revolution mathematics has been one of the most compelling directive forces. It is a force in war and defense, and it is a force in peace. In all human activity it has become indispensable, because it is the discipline by which man magnifies his mind. You are the purveyors of the principles of this subject. It lies in your hands, whether the students who come under your charge shall see this subject for what it is, or shall carry away mistaken notions or unfortunate prejudices against it. If this subject is not in your consciousness as a dynamic one, it will only be the exceptional student who recognizes it to be such in spite of you. You need never apologize for mathematics. It is a living subject—indeed, it is so abundant of life that it vivifies whatever other subject comes into contact with it. The sciences, one and all, depend upon it, and their evolutions are all in the direction of becoming more mathematical. Technology is nothing more than mathematics at the practical level.

In your classes, be alert to the pupil with mathematical ability. Be sure you recognize the ability when it is there, for the most industrious student is not always the most able. And give of yourself what you can to the best students, remembering that there is much reason to believe that their abilities are often much greater than we give them credit for. We know this from experiments such as that of the Ford scholars, who go successfully from the sophomore high school classes into college. Mathematical ability often ripens early. Important parts of the history of mathematics are indeed biographical of precocious geniuses who made significant pioneering advances at high school- and college-student ages.

This is now the place for me to conclude. From what I have said, it should be clear to you that it is something fine to be a mathematics teacher, and that your assignment in the social scheme is a significant one, and one of growing importance.

With algebra begins a new adventure. The first eight years of math have been much the same. They were never too interesting, and were made even less so by the fact that they were "pounded in". Algebra is not only new, but to the student who really learns it, it is interesting and fun. Acts that would have seemed impossible to the student when in grade school, are now performed by him with ease. He becomes sort of a miracle man and he enjoys it immensely. He discovers that his knowledge of setting up equations and working with them can be used for many everyday problems. And in discovering this, he realizes how slight this knowledge of his is, and the importance of enlarging it.

So he takes Geometry. Geometry is completely different from the math that he knows. In this course he learns to think clearly and to organize his thoughts on paper. Each math learned applies to the next one, so the student who becomes interested in exploring this realm continues with the subjects. The third year brings more Algebra and Trigonometry. In the first, he furthers his abilities to work equations and learns new methods. In the latter he learns the importance of accuracy.

After completing the fourth year, the student walks out of this world a much more able person for having been there. He is qualified to go on to higher mathematics in college. His background will enable him to go into important fields of engineering. Even if the student does not intend to study further, he has learned many valuable things. What else can equal a knowledge of clear thinking, accuracy, thoroughness, and organization?—*Carole Cross, Student, Wayne Memorial High School, Wayne, Michigan*

Concept learning in algebra

MAX A. SOBEL, *South Side High School, Newark, New Jersey.*

What does meaningful learning contribute to mastery of mathematical concepts? This study furnishes one bit of evidence which can be used in formulating a hypothesis.

THE IMPORTANCE of an adequate comprehension of basic concepts of mathematics has long been recognized and accepted by educators and emphasized in reports on mathematics education. Thus, the *Fifteenth Yearbook* of the National Council of Teachers of Mathematics states:

> An understanding of the concepts and principles of mathematics is the key to its successful study. To study in such a way that the concepts become clear is the hardest and most significant task confronting the teacher of mathematics.[1]

More recently Fehr considered the importance of the learning of concepts in the entire field of mathematics when he wrote:

> The learning process of going from experience with things, to thinking about things, to abstraction and concept formation, and finally to reorganizing the newly learned concept into the whole structure is, in the field of mathematics, the one that has the most promise of permanence of learning to the solution of quantitative problems. The building of concepts is so necessary if we are to develop mathematics as a way of thinking that will serve us in our various life careers.[2]

The poor presentation of concepts in school instruction has also been recognized by mathematics educators. Breslich has adequately expressed this thought in the area of algebra:

> Not enough experiences are provided to develop real understanding of the concepts of algebra. Literal numbers, signed numbers, symbols of operation, exponents, equations follow one another in rapid succession, although each alone offers a serious difficulty to the learner. Most textbooks introduce all of these concepts within the first 25 to 30 pages. Hence no real understanding is attained by the pupil. The result is confusion and dislike of a subject which is in reality simple if the fundamentals are thoroughly mastered.[3]

Despite the recognition given to the importance of concepts in the field of mathematics, very little research has been undertaken to investigate the method by which these concepts are best learned. The relationship between emphasis on mathematical concepts and performance of fundamental skills is another problem which has likewise been left unanswered.

If the acquisition and understanding of concepts is as important to the learning of mathematics as the literature claims them to be, then it seems reasonable to make the assumption that it is of equal importance to determine whether or not there is a particular method by which these meanings are best learned. This paper describes a study whose aim it was to discover whether or not there is any relationship between the learning of certain algebraic concepts and their method of presentation.

TEACHING METHODS

Two methods of teaching were compared, namely:

A. An abstract, verbalized, deductive method with concepts defined and presented by the teacher, followed by practice

[1] "The Place of Mathematics in Secondary Education," *Fifteenth Yearbook* (Washington, D. C.: The National Council of Teachers of Mathematics, 1940), p. 57.

[2] Howard F. Fehr, "Reorientation in Mathematics Education," *Teachers College Record*, LIV (May, 1953), p. 435.

[3] Ernest R. Breslich, *The Administration of Mathematics in Secondary Schools* (Chicago: The University of Chicago Press, 1933), p. 267.

exercises. Henceforth, this shall be termed the control method.

B. A concrete, nonverbalized, inductive method with students guided, through experiences involving applications, to discover and verbalize concepts. Hereafter, this shall be known as the experimental method.

The first of these two methods is the approach offered by most textbooks and consequently is assumed to be the development followed by most teachers of mathematics, whereas the second method of presentation is the one advocated by those who write of teaching for meaning, with emphasis upon the role of the student in the learning process. Much has been written concerning these methods, with the emphasis in favor of the latter approach. For example, Rosskopf writes:

> In the first place, when a class is introduced to a new concept there must be active student participation in discovering the concept and how to apply it. Such active student participation will depend upon adroit questioning on the part of the teacher and upon his sensitivity to the progress of the class in its exploration of the concept. As soon as students are aware of the concept, they are ready to apply the concept. They are ready for practice problems, if you like, or for an opportunity to solve new tasks that are different from the demonstration examples but require the concept (used here as a synonym for principle or generalization) for their solution. At this stage of progress of the students a teacher must be satisfied with students understanding how to apply the concept; no attempt should be made to have the students state the principle in words nor should the teacher give the statement of the principle in words. One might say at this stage the students work intuitively.[4]

Because of all that has been written similar in nature to the preceding statement, the following hypotheses were formed:

1. The experimental method of teaching will produce results which are significantly superior to the control method on a test which is designed to evaluate certain algebraic concepts.

2. This superiority will also be evidenced on a test which evaluates certain fundamental algebraic skills.

3. Both the concepts and skills will be retained better by the students taught through the experimental procedure.

"Concept" defined

A survey of the literature which has appeared during the past thirty years dealing with concept learning reveals a lack of agreement among psychologists and research workers concerning the nature of this process. The term "concept" has been employed to describe a variety of processes, with each writer insisting that his work is a true study of concept learning and disputing the claim of previous research workers concerning this point. Experimental studies have been relatively simple, generally studying the formation of single, isolated nonsense symbols which have been called concepts. None of these approximate the complex situations in which concepts are learned within the classroom or in daily life.

Although the literature reveals a lack of agreement concerning the exact meaning and nature of a concept, this difficulty may be avoided in the field of mathematics by operationally defining a student's concept of some particular term as the summation of a given set of responses which the student is expected to elicit at some given stage in his mathematical development. That is to say, at any particular level of instruction a list of specific questions may be formed which serves to evaluate the degree of mastery of a certain concept. Possession of the concept is assumed if the student is able to respond successfully to these questions.

The operational approach was adopted to avoid involvement in an abstract discussion of the nature of a concept. The major theme of this report deals rather with an examination of two distinctly different methods by which concepts may be learned.

[4] Myron F. Rosskopf, "Transfer of Training," *Twenty-first Yearbook* (Washington, D.C.: The National Council of Teachers of Mathematics, 1953), p. 218.

Description of the experiment

A group of six schools in Newark, New Jersey, and one in Paterson, New Jersey were secured where there was an interest in co-operating in a mass experiment. These schools covered a wide range insofar as ability and socioeconomic level is concerned. With the help of principals and department chairmen, teachers were secured who were willing to take part in the experiment. Seven algebra classes at the ninth-grade level were chosen, which were to be taught by the control method, with an equal number to be taught by means of the experimental method. Each class was taught by a different teacher with the exception of one instance where a teacher taught two experimental classes. Care was taken to assure reasonable equality between the two groups of teachers insofar as teaching experience and ability were concerned.

Each of the experimental teachers was given a manual of instructions which presented a broad, over-all view of the problem, discussed the several methods of approach to be tested, and provided numerous illustrative examples. The control teachers were given instructions to have their students master a set of specific skills within a given period of time, using the development found in their textbook for this purpose. The experiment took place at the start of the semester and was four weeks in length. It was designed to include the concepts of general number, formula, exponent, and coefficient as they appear in a beginning course in algebra.

Evaluation of the experiment presented a problem in that an examination of tests which were available at the time revealed a lack of any standardized instrument designed to measure algebraic concepts. Since no satisfactory test was available, experimental questions were developed during two pilot studies which were undertaken prior to the mass experiment. These were collected and given to leading mathematics educators to examine and criticize. After revisions had been made, two parallel forms of a test were completed, each containing two parts: Part I of the test dealt with concepts (30 questions), and Part II served to evaluate fundamental skills (20 questions). One form of the test was administered at the conclusion of the period of experimentation with the results shown in Table 1.

Inspection of the groups on the basis of I.Q. indicated that several of the classes

TABLE 1

Means and Standard Deviations of Classes with Respect to Parts I and II of Test Form A

Part I	10.44	9.21	10.57	8.67	10.62	10.26	12.88
s	3.00	2.91	4.14	2.68	4.66	3.74	4.04
Part II	8.28	7.21	7.76	7.22	7.71	7.63	8.19
s	2.86	2.95	3.82	4.18	3.76	3.08	3.29
N	18	19	21	27	21	19	26
			Control Groups				
Part I	10.85	9.62	9.05	8.58	8.91	15.26	16.62
s	3.34	3.87	3.30	3.77	2.90	3.04	3.61
Part II	7.04	8.15	5.88	6.37	6.39	11.37	12.31
s	3.64	3.72	3.14	3.74	3.69	2.81	2.65
N	26	26	19	19	23	19	29
			Experimental Groups				

TABLE 2

MEAN SCORES AND DIFFERENCES OF SUBGROUPS WITH RESPECT TO PARTS I AND II OF TEST FORM A

Group	Mean Control	Mean Experimental	Difference (Exp.—Con.)
Average I.Q.			
Part I	9.64	9.49	− .15
Part II	7.58	6.86	− .72
High I.Q.			
Part I	11.41	16.08	+4.67
Part II	7.88	11.94	+4.06

were superior to the others so that subgroups could be formed. This allowed for the comparison of groups on the basis of both method and ability, all data being considered in relation to the following four subgroupings:

1. Control group, average I.Q. (100)
2. Control group, high I.Q. (110–15)
3. Experimental group, average I.Q. (100)
4. Experimental group, high I.Q. (110–15)

In Table 1, the first four control groups and the first five experimental groups were classified as average I.Q. groups, the remainder as high I.Q. groups on the basis of class means on intelligence tests. The results of each subgroup were pooled and compared when statistical analyses established homogeneity of means and variances within these groups on each part of the test. Table 2 summarizes the results of the comparison.

The hypothesis was set up that no difference existed between the mean scores made by the experimental and the control groups at each of the I.Q. levels; that is, that the true difference is zero. The following limits were established within which a true difference would fall in 95 cases out of 100, the hypothesis of no difference to be sustained if zero is included within these limits.[5]

[5] Use was made of the expression $d \pm 1.96\sigma_d$, where d is the observed difference and σ_d is the standard error of the difference between means, computed as

$$\sigma \sqrt{\frac{n_1+n_2}{n_1 n_2}}.$$

	Average I.Q.	High I.Q.
Part I	−1.12 to +.83	+3.18 to +6.17
Part II	−1.75 to +.31	+2.88 to +5.24

These figures indicate that a true difference of zero might quite possibly occur by chance for the average I.Q. groups, whereas the hypothesis of a difference of zero on either part of the test must be rejected for the high I.Q. groups.

RETENTION OF CONCEPTS

At the conclusion of the period of experimentation, neither group of teachers was given instructions as to methods of teaching, and both teacher groups were left free to proceed in any manner they wished for the remainder of the term. Three months after the original test was given, a test was administered to the same students to determine the effect of the method of presentation upon retention of the concepts being studied. The results are summarized in Table 3.

Once again homogeneity of means and variances within each subgroup was established and the results were pooled on the basis of high and average I.Q. The comparison is summarized in Table 4.

The following limits were computed at the .05 level of significance within which we may be confident the true difference will fall:

	Average I.Q.	High I.Q.
Part I	− .92 to +1.57	+2.40 to +5.13
Part II	−1.14 to + .92	+2.23 to +4.46

It is clear that a hypothesis of a true difference of zero may occur for the aver-

age I.Q. groups, whereas one may have confidence that in 95 cases out of 100 this would not happen by chance for the high I.Q. groups. Thus the null hypothesis must be rejected for the latter group, and the observed difference in favor of the experimental method may not be ascribed to chance but considered to be significant.

Discussion of results

The major implication of this study is that bright students learn and retain certain concepts and skills better through an inductive, concrete, unverbalized teaching approach as opposed to a deductive, abstract, verbalized method of presentation. This is contrary to the method by which concepts are presented in the majority of algebra textbooks where authors present a great many concepts within a short space of time and introduce them quite formally and abstractly. Thus the teacher who plans to develop concepts in accordance with the experimental process discussed in this article generally finds that he cannot rely on the textbook for this purpose but must develop his own materials and methods.

The results of the experiment were unexpected in that previous research studies had failed to indicate any definite relationship between intelligence and the

TABLE 3

Means and Standard Deviations of Classes with Respect to Parts I and II of Test Form B

Part I	13.50	11.14	12.50	11.23	14.42	14.72	13.23
s	3.46	3.27	5.08	3.92	4.18	4.21	3.28
Part II	10.50	10.00	10.94	9.88	10.42	11.78	12.39
s	3.24	3.07	4.58	4.56	3.56	3.49	2.71
N	16	14	18	26	19	18	23
			Control Groups				
Part I	13.56	11.18	11.88	11.33	13.26	17.06	18.24
s	3.21	4.07	3.51	4.26	4.99	2.38	3.01
Part II	10.40	10.64	10.12	9.56	9.39	14.50	15.17
s	4.02	3.91	2.54	3.69	4.02	2.00	2.02
N	25	22	17	18	23	16	29
			Experimental Groups				

TABLE 4

Mean Scores and Differences of Subgroups with Respect to Parts I and II of Test Form B

Group	Mean Control	Mean Experimental	Difference (Exp.—Con.)
Average I.Q.			
Part I	12.01	12.34	+ .33
Part II	10.29	10.04	− .25
High I.Q.			
Part I	14.05	17.82	+3.77
Part II	11.58	14.93	+3.35

method of learning concepts, other than to say that the ability to attain concepts is correlated positively with I.Q. This is probably due to the fact that many of these previous studies were undertaken at the college level where only the superior student was involved.

In the present study only two methods of teaching were compared. It is possible that a third method can be formulated which would produce results superior to either of the two methods investigated. Again, the time limit was rather short. For the average I.Q. groups, a longer period of experimentation may have produced results which would favor one of the methods; the results of the high I.Q. groups may not have evidenced the superiority for the experimental methods. Perhaps bright students can learn equally well by either method, but need more time to do so by the control process.

It would be interesting to perform a similar experiment, using other than algebraic concepts, and including a third group composed of students of low I.Q. At present no evidence has been presented as to the best method for developing concepts for students whose I.Q. is below 100. Is it possible that the control method might be favored by the low I.Q. group? If so, this would be contrary to the method by which educators advise handling this particular group.

The two methods of teaching investigated in the study described in this report resulted in no real difference in learning for average I.Q. groups. This would seem to imply that for this group one method may be used just as well as another. Nevertheless, consideration should be given to the experimental method of teaching in that it is the one which is claimed to be psychologically superior by mathematics educators, when considered from the point of view of teaching mathematics for meaning. The brighter students of the class will profit from this procedure, whereas the average group should fare no worse than if taught by the other method.

All of the recent literature in the field of mathematics education speaks of the importance of learning in a concrete setting with the student left free to make discoveries and form generalizations as is so often required in daily-life activities. This, the experimental method attempts to do. Although it did not make for better learning in all groups in the present study, it does strive to develop certain attitudes and appreciations which are of importance in the meaningful learning of mathematics.

The evidence provided by a single study has numerous limitations and can not settle controversial issues with any degree of finality. The value of such an experiment comes rather from the addition of evidence, the accumulation of which may eventually point toward definite answers to certain key questions. It is also hoped that the results of the study described in this article will serve to encourage and stimulate further research in this area by others.

If the formation of concepts is as important as mathematics educators claim, then it is likewise of importance for mathematics teachers to search for further evidence on the subject. The entire area of concept learning in mathematics is a virgin one awaiting detailed scientific investigation to provide carefully documented evidence as to the nature and process of concept formation.

Mathematics in New Zealand Schools

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While courses may differ, people and their learning problems are alike the world over.

THE PURPOSE of this article is to give American readers some idea of the nature of mathematics teaching in the schools of the Dominion of New Zealand. Cursory examination of United States textbooks and articles in NEA magazines leads this writer to believe that the problems in the two countries are somewhat similar. Perhaps this interchange of information will be interesting for that very reason.

Assuming that few readers will know a great deal about our school system, it seems advisable to give some account of our practices as a general introduction to the main theme of this article.

The New Zealand educational system is almost completely homogeneous. There is a single Education Department under a director who is a public servant responsible to the Minister for Education in the Parliamentary Cabinet. The Education Department delegates many of its functions to ten educational boards and 50 or 60 post-primary boards of managers; but in reality the department governs all public education in the dominion, both primary and post-primary, and even influences, greatly, the university programs. Standards are maintained at the same level throughout the dominion by the publication of national syllabuses of instruction and through the inspectors who are departmental officers although domiciled in the local education boards. Identical philosophies of education are advocated in all parts of New Zealand through the strength of financial centralization in the Education Department. Yet the widest measure of freedom of actual practice exists, and the two-way traffic in ideas between center and periphery is consciously fostered by all agencies in education.

Teachers are organized in two national bodies—the New Zealand Educational Institute and the Post-primary Teachers' Association. The NZEI is the only primary school teachers' society, but has some 300 post-primary members among its 8500 members. The PPTA is solely post-primary. It has some 2000 members. Although strong in branches, each of these societies of teachers, like the employing authority, is highly centralized in ideas, policy, finance, and bargaining power.

Primary schooling commences for almost all children at five years of age, actually on the fifth birthday in most cases. Two years, divided into four half-years, generally suffice for the primers or infant classes (P1 to P4), and four years for the standards (S1 to S4). Then come Forms One and Two which are still primary, but some 25 per cent of the children are housed in separate intermediate schools. Thus, New Zealand children have a total of eight years of primary schooling, corresponding roughly to grades one to eight in United States' schools.

The post-primary education, accessible without examination to all children, follows in Forms Three to Upper Sixth, a four-year course, but with no obligation to remain in college after the fifteenth birthday. Only from the seventh till the fifteenth birthday is schooling compulsory. Nevertheless, most go to school from the

fifth birthday to beyond the sixteenth. After a minimum of three years at post-primary school, a student may take the School Certificate Examination. The papers and conditions are absolutely uniform throughout the country. The student presents English as the sole obligatory subject plus three others selected from thirty-four subjects. These include several foreign languages, pure and applied mathematics, natural and physical sciences, history, geography, and many practical subjects like embroidery, woodwork, horticulture, technical drawing, etc.

After a minimum of four years at college or high school, the student may present himself for the University Entrance Examination. Here, again, the student presents English and three other subjects from a list similar to the preceding. This qualification may be awarded without examination (accrediting) or may be passed at a uniform examination controlled by the University of New Zealand. Upper Sixth forms exist for those students who are going to the university and who wish to advance further in their studies toward this end. Most university bursaries (scholarships) go to these students.

Education is without fee throughout this long stay at school, which varies from 10 to 14 years. It is subsidized by family benefit (10 shillings a week—about $1.50) from birth till the end of the year in which the student turns eighteen. There are also many other concessions and bursaries. University education is virtually without fees and is heavily subsidized by a wide bursary system.

Nonstate schools exist throughout the country. Most are denominational, mainly Roman Catholic, and receive very minor financial, but considerable aid, from the state. About 12 per cent of the primary children and 15 per cent of the post-primary children attend these private schools, but the basic education, classes, textbooks in most subjects, standards, and the public examinations taken are identical with the state system. The achievements in these private schools are certainly no higher than that of the public schools.

Mathematics, at primary school level, is arithmetic. In the classes, emphasis is on experience with quantities, numbers, weights, and measures, with little formal work. The number combinations up to 20 are sometimes taught at this level. Then mechanical skill is emphasized, passing to "social" arithmetic and elementary problem-solving later. By the end of Form Two, the pupil is expected to be able to handle successfully numbers to one million, fractions, decimals, and percentages, elementary mensuration, simple proportion, and the associated problems found in daily life. Some simple pictorial graphing is often included in the course. The textbooks used are a series published by the Education Department's own Publications Branch. These are frequently supplemented by sets, almost exclusively British in origin, bought by the schools themselves. Methods are laid down in the syllabus and in the textbooks, and varied by the teachers as they see fit.

At the post-primary level mathematics is at two quite distinct levels—Core, which all must take, and Full, which is optional and includes the Core. The Core Syllabus aims at solidification of the primary schoolwork and its extension in mensuration, applications of percentages, the metric system, and simple graphics. The Core also includes very elementary algebra and geometry.

This course occupies from an hour and a-half to three hours per week for each of three years, depending on individual school time-tables. This does not allow anything like enough time to make mathematics an option for School Certificate. With boys in technical high schools, or in technical courses in ordinary high schools, the work may continue into the fourth year. Girls are more likely to take token mathematics courses in the first three years at college.

Full mathematics as an examination subject is quite different. Within three years the student is expected to acquire skill in these subjects to the stated level: Arithmetic—to compound interest, logarithms, difficult examples in mensuration including volumes and costs; Algebra—to solutions of difficult quadratic equations and remainder theorem, and the correlative graphs; Geometry—to similar triangles, ratio, and proportional division; and Trigonometry—to solutions of triangles including sine and cosine rules, functions of angles to 180 degrees, and simple identities. Time allotment for the Full course varies among schools, averaging between four and six hours weekly. Perhaps 30 per cent of the children take such a course. Applied mathematics is a separate subject under the name of applied mechanics and is taken in few schools outside metropolitan technical colleges. In these schools geometrical and technical drawing are other allied subjects that lead to examination level.

Beyond School Certificate, mathematics is a serious subject mainly for boys preparing for science, engineering, and architecture at the university. The subjects are: Algebra—including surds, progressions, binomial theorems, variations, theory of logarithms; Geometry—to Ptolemy's Theorem and solid geometry in its elementary stages; Trigonometry—to functions of submultiple angles, functions of angles of all magnitudes, identities, and equations. The standard is very much higher than that for School Certificate and yet so much below the first stage at the university that most students need a preuniversity year in the Upper Sixth Form before having any chance of coping with the later work. Six hours a week would be about the usual allocation to mathematics in the university entrance year. At all stages of Full mathematics work, the student will have homework taking at least as much time as he has on the school timetable.

Generally speaking, the teaching and achievement in classes preparing for Full (i.e., examination) mathematics is adequate and often at a high level. Many who sit the examinations are below the required standard because they should never have taken the option. The shortage of qualified teachers in mathematics and science is very great indeed. Many schools find it very difficult to meet the requirements of their best students. Available teachers of ability take the top classes. The obvious result is that the standards in the so-called Core classes are too frequently below what is desirable. These classes are the training ground for tradesmen (among other occupations), so the need for sound mathematical training is particularly important. New Zealand tradesmen need all-round skilled abilities, as we have nothing like the large-scale industry with the consequent fine division of labor found in the United States or Britain.

To overcome this sad lack of really good training at a level below examination, yet above the rudimentary, a wide system of daytime classes for apprentices has gradually grown over recent years. These youths, from 15 to 20 years of age, spend the equivalent of three working weeks annually at technical colleges, in normal working hours and on full wages, catching up on technical training for their trades and such theoretical work as mathematics applied to industry. As it is unlikely that enough graduates in mathematics and science will come to the teaching profession with government and industrial demands at their present level, such teachers are being "made" in many schools by dint of devoted effort on their part and generous guidance from better qualified colleagues. In these two directions may lie the solution of a serious problem for our young but vigorously growing industry.

New Zealand vocabulary, social setup, and educational traditions and practices make most British textbooks very effective in the teaching of Full mathematics. The most-used authors are Englishmen

like C. V. Durell, A. W. Siddons, H. S. Hall, H. J. Larcombe, W. G. Borchardt, and R. T. Hughes. The time-honored textbooks like Baker and Bourne, Hall and Stevens, Barnard and Child, and Channon and Smith, never quite go out of favor. There is no difficulty here. The troubles come in the other field, that of Core mathematics. Here the need is for material and methods stemming from the differences between us and the United Kingdom rather than from the similarities. As an aid to the teaching of Core mathematics, the Education Department has sponsored a locally written textbook. It has had only a few months in the schools, so no valid opinion based on experience is yet possible. Perhaps modifications in the light of experience will make this book the aid to teaching demanded by the paucity of qualified, able teachers.

Some figures may help summarize and conclude this cursory attempt to convey something of the mathematical setup in New Zealand schools. The total population is 2,100,000 spread over 100,000 square miles. More than 300,000 children are in state primary schools; over 70,000 in state post-primary schools; about 11,000 are in universities; and another 2800 in teachers' colleges. Over 40,000 children are in registered private primary schools, and over 12,000 in similar postprimary schools. Annually, more than 13,000 sit the School Certificate Examination, and 4000 sit the University Entrance Examination. Generally, about two-thirds pass in each case. Education will cost more than 30 million pounds this year, which is almost 90 million dollars, and is between 11 and 12 per cent of the national budget. The country has such a successful policy of full employment that more than 25,000 jobs go unfilled at any one time. Nobody knows the number of mathematics- and science-trained people who are needed to fill vacant positions in industry, the municipalities, and government posts. Teaching could absorb, perhaps, from 300 to 400 such graduates at once.

Short multiplication

It is impressive for any mathematics teacher to have a few entertaining methods to present for solving problems. Indeed short multiplication cannot compete with today's computing machines; but it is good to know it.

Suppose we would like to multiply 7251 and 3522. We write one of these numbers under the other as follows:

$$\begin{array}{r} 7251 \\ 3522 \\ \hline 25538022 \end{array}$$

The product is written from right to left without use of partial products. To find this product we take care of one digit at a time.

I. The unit's digit is found by multiplication of units: $(2)(1)=2$.

II. The ten's digit is obtained by multiplying units and tens: $(2)(5)+(2)(1)=12$. Therefore the ten's digit is 2 and one remains to be added to the hundred's.

III. For the hundred's digit we must multiply units by hundreds and tens by tens and add to it the one left from part II, namely $(2)(2)+(1)(5)+(2)(5)+1=20$. So 0 is the hundred's digit and 2 is left to be added to the thousand's.

IV. For the thousand's digit we multiply units and thousands, tens and hundreds, and add the 2 left from III: $(2)(7)+(1)(3)+(2)(2)+(5)(5)+2=48$. The thousand's digit is 8 and 4 is left for ten thousand's.

V. For the ten thousand's digit we multiply tens by thousands, hundreds by each other, and add the 4 left from IV: $(2)(7)+(5)(3)+(2)(5)+4=43$. So 3 is the ten thousand's digit and 4 is left for hundred thousand's.

VI. The hundred thousand's digit is obtained by multiplying hundreds by thousands and adding 4 which is left from V: $(5)(7)+(3)(2)+4=45$. Here the hundred thousand's digit is 5 and 4 remains for the millions place.

VII. Finally millions is found by multiplying the thousands together and adding 4 remaining from VI: $(3)(7)+4=25$. Consequently the product is 25538022.

This process can be used for multiplication with more digits, and as it is practiced a few times, the superiority of the method is easily seen.—*Ali R. Amir-Moez, University of Idaho*

The technical manpower shortage—an answer from the high school

THEODORE S. COOPER, *Denver Public Schools, Denver, Colorado.*

A high school teacher with a variety of experiences in counseling, curriculum construction, and advising students gives his reactions to some of the pressures put on the high schools to modify their programs.

DURING THE PAST few years a number of articles have appeared in magazines, newspapers, and professional journals concerning the growing technical manpower shortage.[1] Authorities from industry, business and various professions have produced, collected, analyzed, reanalyzed, and commented upon an impressive number of statistics which show the graveness of the problem and *why* it exists. Far too many of these authorities reach the conclusion that much of the fault lies with our public high schools; more specifically, with inadequacies of the high school science or mathematics programs.

Without question, the United States does face a serious present and future shortage of technicians and engineers; however, the problem will not be solved by trying to place the blame or shifting the responsibility to any one particular educational or societal agency. Everyone has a stake in the outcome; hence, everyone must share equally in the responsibility for what has happened in the past and what is to be done at present and in the future. Far more will be accomplished if authorities from colleges, industries, and the professions will direct their efforts toward correction of shortcomings within their own ranks rather than devoting their time to criticizing the secondary schools.

A major hurdle to overcome is the lack of communication between and among the various agencies. College professors, industrialists, businessmen, writers—in fact, nearly all high school graduates—make a common basic assumption that does much to close the channels of communication to and from the secondary schools. They are inclined to feel and say: "What can we gain by a visit to the high school? After all, we attended high school and we *know* exactly what is going on there." Such a statement is utter nonsense! At the time these alumni were attending high school, they were little interested in, or aware of, the technical problems involved in the teaching or administration of their school.

College professors are major offenders

[1] For example:

The statistical survey of 104 public school systems made by John Latimer of George Washington University for *Time* Magazine, December 13, 1954.

Dorothy Thompson, "Our Achilles Heel. Soviets Might Win School Race," published on various editorial pages throughout the nation by Bell Syndicate, Inc., January, 1956.

"Freedom's Need for the Trained Man," a mimeographed copy of the remarks made in an address by Lewis L. Strauss, chairman, United States Atomic Energy Commission to the Alva Edison Foundation Institute, Washington: United States Atomic Energy Commission, November 21, 1955.

Joel H. Hildebrand, "Intelligence Is Important," *Chemical and Engineering News*, XXXIII (October 17, 1955), p. 4414.

Leonard G. Yoder, "Mathematics and the Technical Manpower Shortage," THE MATHEMATICS TEACHER, XLVIII (December, 1955), pp. 548–51.

in this respect for they "not only have attended high school, but *trained the teachers* who teach in the high schools." This is, to a large extent, also utter nonsense. Very little of any professor's lectures or philosophy will ever appear in a high school classroom. What does appear is material learned long ago from another high school teacher, or accumulated through individual study and experience, and presented through techniques personally developed or learned by observing and/or working with high school teachers.

The seriousness of this communication difficulty and its effects become apparent when the occasional outsider visits the high school. Then, and only then, can we show the "noninformed" conclusive visual evidence as to why the high school is not producing up to his expectations.

What more convincing proof could there be than to visit all the mathematics courses in a particular high school, thereby noting that, for every class in elementary algebra or formal mathematics, there is likely to be one or more classes, usually titled "basic" or "general," containing students who have not learned or *cannot* master even the elementary processes of addition, subtraction, multiplication, division—or, worst of all, those "terribly complex things" called fractions and decimals.

Even more enlightening, perhaps, would be the experience of observing the large percentage of youngsters enrolled in second or third year formal courses who are completely unqualified for the course —who are there in "body only." These are the youngsters the colleges will soon point out as examples of failure caused by poor preparation in high school, never considering that, due to various factors, the task of training these individuals is as "hopeless" at the high school level as it will be later on at college. These are the victims of the American misconception that "freedom of education" implies that the same education is good for all, regardless of abilities or interests.

Finally, the outsider should observe that changes in high school curriculums and activities have occurred so rapidly in the past few years that he will be a rare alumnus if he can find anything similar between the high school he knew and the one he may now visit.

Quality needed—not quantity

It is a common American philosophy that quantity will solve nearly any problem. If one wants to make people happy, one gives them more television sets or more cars. To be successful, one must make more money than anyone else. To win a war, one must have more bombs, more airplanes, and more engineers than the enemy. Produce more, and one is sure to win.

We still have a tremendous store of natural resources, so perhaps we can produce more material resources than our enemies; however, our stockpile of human beings capable of being shaped into engineers and scientists can never match that of Russia and its satellites. How, then, can we hope to survive? Germany gave us an answer in World War II: efficiency. Utilize every potential technician to the maximum of his ability and, even more important, use him only in positions where he can apply his ability and training to the fullest extent.

The recent publicity campaigns by industry and the colleges to interest more young people in science and mathematics, while showing good intentions, may have intensified the problem rather than helping to solve it. This seems due to the usual lack of communication between high schools and other agencies which have determined, by the American philosophy of quantity, that to have more technicians we must encourage more and more students to enter high school courses in science and mathematics.

For as long as the American high school has existed, science and mathematics teachers have worked to refine methods of selecting, encouraging, and training students who appeared to have scientific

potential and interest. In the past few years, increasing importance has been attached to these procedures, partly due to the greater societal interest in science and partly due to other factors.

Giving encouragement is important, but it should be given to "the right individuals at the right time and directed toward achievement of the right goal(s) for these individuals." Little is gained by trying to encourage the wrong individuals to do something which, for them, is "impossible."

A second and more serious counseling procedure exists, over which the classroom teacher has little control. This is the periodic overenrollment in mathematics and science classes caused directly or indirectly by the publicity campaigns of overenthusiastic pressure groups outside the school who suddenly find themselves in need of greater numbers of technically trained personnel. In their rush for quantity, these groups seem unaware of or unconcerned about the problem they create in the secondary school in the form of large numbers of badly qualified students demanding admission to mathematics and science classes. Do these groups really believe that the quantity gained is worth the quality sacrificed?

The quantity attitude does not exist just in industry or private business. It is equally prevalent in school policy and philosophy. For example, the high school mathematics teachers of Denver were recently "commended" because a city-wide survey[2] disclosed that, considering total school enrollment, enrollment in formal mathematics courses had increased from 34.8 per cent in 1950 to 37.5 per cent at present—thereby showing that Denver schools were really doing something to help overcome the technical-manpower shortage problem.

[2] An informal survey made by the Department of Instruction, Denver Public Schools, and reported to members of the K-12 Mathematics Committee, Denver Public Schools, Denver, Colorado, April, 1956.

Certainly, the interest and effort being put forth by Denver mathematics teachers will do much toward helping to solve this problem, but it is the opinion of this author that the real measure of success will not be found in the greater numbers enrolled in classes. Rather, would it not be preferable to measure the success of solving the problem by the *quality* of the technical personnel being produced? To state the question precisely: Will it serve our country best to have a small number of experts—or shall we place our chances for survival in the hands of many amateurs?

Also, it should be pointed out that statistics such as the enrollment figures cited should be taken "with a grain of salt." They are summary statistics compiled by the central office of a large school system rather than on-the-spot statistics compiled at the point of measurement. Regardless of justification, individual schools and their various departments are inclined to revise their tabulations of class achievement so that their particular part of the program looks good to their superiors. The central office, being responsible to the school board and to the general public, makes some revisions of its own. The ultimate statistics and their interpretation could provoke nothing but humor if they were not taken so seriously by so many people.

For example, in the Denver survey,[3] the results eventually reported show "a healthy increase in enrollment in formal mathematics classes accompanied by a decrease in the drop-out and failure rate." Why, then, do the individual high schools have statistics similar to those found in the following three-year study made by one high school mathematics department?[4]

These were the figures given concerning enrollment and success in formal mathematics courses:

In 1954, 200 (37 per cent) of 535 stu-

[3] *Ibid.*

[4] "Manual Looks at Its Mathematics Program." A report to the principal by the mathematics department, Manual High School, Denver, Colorado, June 1, 1956.

dents were in algebra or geometry courses; 20 or 10 per cent failed.

In 1955, 218 (60 per cent) of 362 students were in algebra or geometry courses; 76 or 35 per cent failed.

In 1956, 321 (53 per cent) of 605 students were in algebra or geometry courses; 145 or 45 per cent failed.

Note that the net number passing was 180, 142, and 176 in the years 1954, 1955, and 1956 respectively—indicating that not only does most of the added enrollment fail but it probably takes with it part of the original number of potential passing students.

Why, then, are we claiming growth through quantity? Is it that we are unwilling to face additional attacks by the ever present school critics who can sometimes be convinced by enrollment increase statistics? Or are we, perhaps, unintentionally looking in the wrong direction in seeking a solution to our problems?

Needed: more communications, fewer statistics

Much could be done to resolve many of our educational problems if a genuine effort were made by all agencies concerned to establish a two-way communication system between themselves and our high schools. It is not enough for our colleges to establish summer science-teacher's programs or other similar activities to "bring the high school teacher up to date." This is too much like a father calling his adult son, and only child, home to give him advice on how to raise a family—after the birth of the son's fifth child. Rather, the college professor should come to know that he can gain a great deal by exchanging visits to the college by high school teachers for his own occasional visits to the high school. For one thing, he might gain increased knowledge of *how* to teach, for, though he may have absorbed more subject matter of a certain kind than his contemporary in the high school, a majority of high school teachers are far ahead of him in the knowledge of how to counsel and teach a very diversified student body.

Leaders of business and industry have also maintained a major roadblock to building a two-way communication system. This is represented in their attitude toward, and often spoken criticism of, the high school teacher as an economic failure in our society and, as such, a person who could offer them little in the way of information that would be valuable. When asked to visit the schools they so bitterly condemn, these leaders often refuse with the comment that they "cannot waste their precious time in this fashion."

A successful communication system to and from the high schools might have prevented the widespread dissemination of misleading statistics that are continuing to appear in newspapers, magazines, and other publications—statistics which are causing increasing numbers of unqualified students to enroll in formal high school mathematics courses.

A popular statistic in many of the published articles is a table, abstracted from a biennial survey of the Office of Education showing percentage enrollments in algebra and geometry classes from 1890 to the present.[5,6] One author[7] points out that figures in this table are revealing. Fortunately, for him, he does not go on to explain exactly what the figures reveal, but it would be a very unusual reader who could correctly infer that enrollments are growing even though the percentages in the table show large decreases from 1890 to 1953.

Perhaps this author, and the many others who have used this table for the same purpose, do understand that our school population has changed drastically from 1900 to 1950 and that it is perfectly

[5] *Biennial Survey of Education in the United States 1948–50* (Washington: Federal Security Agency, Office of Education, 1951), Chap. 5, p. 107.

[6] Estimate for 1953 based on *Mathematics* in Public High Schools, Bulletin No. 5, Washington: U. S. Dept. of Health, Education and Welfare, Office of Education, 1953.

[7] Stephen Romine, *Math and Science Enrollments Up or Down? How Are They in Your High School?* (Boulder: University of Colorado, College of Education, February, 1956), p. 2.

normal and reasonable for enrollment in algebra to drop from 56 per cent to 27 per cent and geometry from 27 per cent to 13 per cent during this period—but they do not enlighten the casual reader with this additional bit of information. Nor has any writer pointed out that only about 10 per cent of the 14–17-year-old population attended school in 1900 while nearly 70 per cent attends at the present time. In other words, rather than a decrease in the total 14–17-year-old enrollment in algebra from 56 per cent to 27 per cent, there has been an *increase* from about 6 per cent in 1900 to nearly 19 per cent in 1953. Likewise, geometry enollment has *increased* from about 3 per cent in 1900 to over 9 per cent in 1953.[8]

Mr. Industrialist wants more engineers so he uses the figures in the table to promote a publicity campaign designed to interest more students in mathematics. Of course, Mr. Average Citizen interprets these statistics to mean that the United States must have more students enrolled in algebra immediately, so he goes to the high school and insists that his son, who has just begun to master subtraction in basic mathematics, be transferred to algebra. That the boy will fail the course does not matter. His father has fulfilled his patriotic duty of building up the algebra enrollment.

Businessmen, college professors, average citizens—all should make periodic visits to the high schools to observe through the years the changing structure of the population and what is being done to meet it. If this were done there would be less criticism of the schools because all of the graduates are not master craftsmen in algebra, geometry, physics, chemistry, reading, writing, and penmanship. There would be an understanding that it is impossible to produce the same or better finished products when the raw materials become progressively poorer in quality.

There would likewise be an understanding that nothing is to be gained by public insistence that schools could be improved by making classwork more thorough and difficult. If the critics would observe in various classrooms, they would soon know that students are now better prepared than in years past. They would also observe large numbers of students, completely foreign to the schools they attended in the past, who are struggling to fit into work beyond their abilities—and failing. They would see their own children, looking far more immature than children of the same age in the classrooms of their youth—more immature because our present-day society and its parents all contribute to delayed maturity.

A visitor to our schools could not help being impressed by the tremendous number of additions to the curriculum and activities he once knew—all in competition with the traditional courses, due to a sharing of time and divided attention.

Finally, there is an all-important change—wherein a boy is no longer considered a "sissy" if he plans a career in art, music, teaching, or something similar rather than planning to be the Horatio Alger variety of "All-American engineer." Today's students are more free from prejudice and coercion in their selection of a career. Also, in many schools, they can plan a career more intelligently with the aid of trained vocational counselors and guidance personnel.

If critics of the high school would pay the schools a visit, they would see these factors and many others at work. Then, and only then, would they adequately understand the framework from which they must work if they sincerely desire to help in the solution of such problems as our dwindling supply of technical manpower.

What can be done?

If little is to be gained by encouraging additional students to enter high school

[8] Practical Mathematics enthusiasts might use this table to show their progress in de-emphasizing formal courses in mathematics. Specifically, all these interpretations will probably be misleading as are most statistics based on non-equated populations.

mathematics and science courses, what can be done to increase our supply of technically trained personnel? From the viewpoint of the high school, at least three major points of attack seem feasible: (1) alteration of high school policies and procedures; (2) alteration of college policies and procedures; and (3) improvement in financial-aid procedures.

First, the high school should re-examine its own role with respect to the potential technicians already enrolled in preparatory courses. If mathematics and science teachers, with the aid of counselors, can establish that some particular student has both scientific ability and interest but is failing to pass courses required for college entrance due to immaturity or other factors, it may be possible to establish procedures which will keep open the "road to the future" for this student. For example, it might help if communications were established with the college of the student's choice so that an exchange of information about the particular student might occur.

If the selected college will accept the student, regardless of his high school achievement—and is prepared to help him from this point forward—the chances for his success are good.

If the selected college is unwilling or unprepared to accept the student on such terms, his counselors and teachers should help him select another college.

As implied in the preceding paragraphs, there is a great need for development and expansion of the counseling and placement programs in high schools. A real attempt should be made to explore the personality and interests of each individual student, to give him facts and information which will help him to select, more intelligently, the career best suited to his particular qualifications.

For decades, engineering schools have been loaded with students who entered with no concept of the demands or qualifications necessary for success. Large percentages of every class have failed to have withdrawn to transfer to other classes on the campus, sometimes due to a lack of ability, more often because they discovered they "didn't belong." How many times has the familiar student cry been heard, "I had no idea there was so much mathematics in engineering. I never cared much for mathematics, anyway. If I had known what was going to be in this course, I wouldn't have taken it."

Much of this misplacement might have been prevented by adequate counseling in the high school. Much of it will be prevented in the future, and it should be noted that such counseling will probably bring about a decrease in the number of students leaving high school to enter technical schools but the net result will be fewer misfits in college classes.

A second point of attack is that of giving increased financial aid to high school graduates planning to go to college.

Last year, the Office of Education estimated that about 17,000 scholarships or aid plans of various types went begging, primarily because there were "no takers." Certainly, there were no applicants for many of these scholarships for reasons such as student apathy or failure to achieve contact between the giver and the recipient. Less publicized was the fact that many scholarships were not awarded, although there were many applicants, because the givers were too particular. Too many of the scholarships specified that applicants must be "all-around good students with high intelligence and unusual mathematics ability." Too few were willing to settle for the next best.

In many instances, there was competition among various agencies to find and provide for the relatively small number of gifted individuals. There was little to offer if the student was only well-qualified. Somehow the fact was overlooked that many of the gifted are quite capable of shifting for themselves financially. Their abilities are such that they can easily work their way through college plus maintaining excellent scholarship records. The some-

what less qualified, on the other hand, will succeed in college only if they can devote most of their time to their studies. These are the students who need a maximum of financial aid.

To avoid the overlapping of aid programs and to assure a maximum return from funds provided, a growing number of individuals and organizations are beginning to utilize the procedure of allowing high school officials to select recipients for such funds in respect to real needs. It is suggested that this procedure and others of a similar nature be explored more fully before large quantities of money are expended which will do little to increase the actual numbers of trained technicians. If industries really want to increase the number of their technicians, they will find it imperative to aid students with lesser qualifications in addition to the aid already being given to the very gifted.

From the viewpoint of the high school, it would appear that colleges and universities are in a position to do a great deal toward overcoming the manpower shortage problem.

Taking their inspiration from programs being developed at such schools as Cornell University, colleges can create mathematics and science courses which will seek out and help to overcome the deficiencies of certain students.[9] Rather than adopting the snobbish or impractical attitude that they "will not fool with students who did not make good in high school," institutions of higher learning must come to recognize that such factors as "extended immaturity" are making it impossible for the high school to prepare students for college as adequately as once was possible. To cope with this condition, colleges will need to develop more flexible mathematics programs.

Even more than in the high school, there is a great need at the college level to develop extensive programs in counseling, guidance, and placement. While many colleges have developed a fairly efficient system of specialized services for this purpose, there is a need to extend these services directly into the classroom. This can be done only by insisting that classroom teachers, especially at the lower division level, have both sufficient training and experience to render these services to their own students. (This does not mean services limited to being able to advise the student on what course to take next!) Such a recommendation may be classified as "ridiculous" by certain college professors because it implies that something beyond the manipulation of their sacred subject matter must occur in their classroom. Sometimes, it might even become essential that subject matter be sacrificed for the student—absolute heresy.

Finally, we must reiterate the time-worn plea to put experienced *teachers* into college classrooms, especially in the beginning courses. Give these *teachers* full-time teaching and counseling responsibilities free of all research and publishing requirements (unless they are concerned with improving teaching techniques and materials for their own classrooms). And, give these teachers *equal compensation and equal recognition*. It is conceivable that this single change could produce an adequate supply of engineers and technicians for now and in the future.

Concluding remarks

In conclusion, it should be pointed out again that there are two broad areas wherein real progress can be made toward overcoming the problem of a technical manpower shortage:

First, all agencies concerned with the training and employment of such persons should strive to improve training and placement procedures within their own confines.

Second, improved systems of communication should be developed among these various agencies thereby conserving time, energy, and money.

[9] Harrison Geiselmann, "Mathematics Deficiencies of College Freshmen," The Mathematics Teacher, XLIX (January, 1956), pp. 22–25.

More on the cutting of squares

MATHEMATICS STAFF OF THE COLLEGE, *University of Chicago, Chicago, Illinois.*

This article continues a discussion that began in the May 1956 issue of THE MATHEMATICS TEACHER, *and also provides material that supplements an article with the same title in the October 1956 issue of* The Mathematics Student Journal.

OUR FIRST PAPER[1] on the general theme of cutting squares gave solutions of Problem I and its converse, Problem II:

Problem I. Given three congruent squares, to transform them into a single square (i.e., to cut them into parts and rearrange these parts in such a way that a single square results);

Problem II. Given a square, to transform it into three congruent squares (i.e., to cut the square into parts and rearrange these parts in such a way that three congruent squares result).

Problem I was also solved in *The Mathematics Student Journal* for April, 1956, so that our first paper had the double purpose of introducing some novel problems in elementary geometry and of supplying some general mathematical background for the *Student Journal* article.

The present paper is a continuation of our previous paper, in the sense that we deal here with the same general subject matter and for the same double purpose. However, what follows is self-contained and can be read without knowledge of our first paper. It is necessary only to know that "to transform" means "to cut into parts and rearrange same"; that two polygonal figures of possibly different shape are "equivalent" provided they have the same area; and that the general problem of transforming one polygonal figure (or several such) into another equivalent one requires a solution in two stages: a *construction* stage (which sees the determination of certain straight segments in the given figure) and a *proof* stage (which sees it established that the parts resulting from a cutting of the given figure along the constructed segments do, in fact, reassemble into the desired shape). *Presentation* of a solution puts the construction first and the proof second; *discovery* of a solution naturally intermingles them.

In the present paper, we shall solve the following four additional problems:

Problem III. Given two squares (congruent or not), to transform them into a single square;

Problem IV. Given a square, to transform it into an equilateral triangle;

Problem V. Given an equilateral triangle, to transform it into a square;

Problem VI. Given a (non-rectangular) parallelogram, to transform it into a square.

Problem III is the problem solved in the October, 1956 issue of *The Mathematics Student Journal.* Problem V is evidently the converse of Problem IV. Problem V was posed as problem A at the end of our first paper, and Problem IV was posed there as problem B. Regarding problem C posed at the end of our first paper, it is readily solved once our solution of the present Problem III is at hand.

[1] "A problem on the cutting of squares," THE MATHEMATICS TEACHER, XLIX (May 1956) pp. 332–43.

The present paper is for the most part based on a recent Russian book, *The wonders of the square*, by Kordiemski and Rusalev.

Solution of problem III

In Problem III, we are asked to transform two given squares (congruent or not) into a single square, i.e., by straight cuts so to divide the two squares into parts that upon rearrangement these parts form a single square. Our solution of this problem is presented in two stages: a *construction* of the cutting lines in the two squares, and a *proof* that the resulting parts reassemble into a square.

Let $ACKL$ and $BFGC$ be our two given squares. Recalling Pythagoras' theorem,

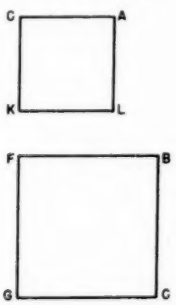


Figure 1

let us arrange these two squares in the familiar relation of Figure 2. Pythagoras' theorem tells us that the area of square $AEDB$ equals the sum of the areas of squares $ACKL$ and $BFGC$. Hence, we have an easy way to construct the square with the desired area. However, this procedure does *not* yield a solution of Problem III because it provides no information as to how our given squares may be cut up and reassembled into square $AEDB$.

The solution we now present of Problem III may also be viewed as another proof of Pythagoras' theorem, and Problem III may be regarded as another motivation for that theorem.

Given squares $ACKL$ and $BFGC$, our *construction* of the cutting lines in them

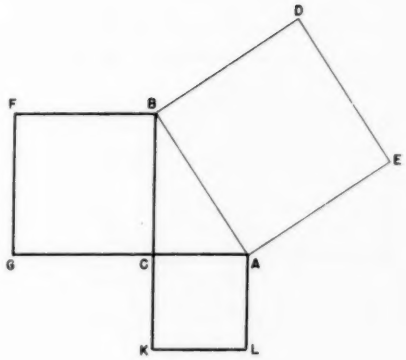


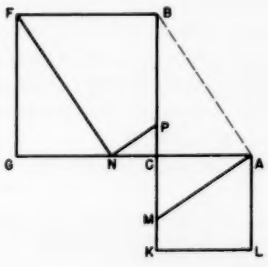
Figure 2

runs as follows: (*a*) Arrange the given squares so that side AC is an extension of side GC and side CK is an extension of side BC, and draw segment AB (note that triangle ABC is therefore a right triangle); (*b*) draw the segment AM perpendicular to AB; (*c*) draw segment FN parallel to AB; and finally, (*d*) draw segment NP parallel to AM. The steps in this construction are shown in Figure 3.

If our given squares $ACKL$ and $BFGC$ are now cut along these segments (viz., square $ACKL$ is cut along AM, square $BFGC$ is cut along FN and NP), the resulting five parts can be reassembled into a single square.

The last stage of this solution consists in a *proof* that these five parts can be arranged in a square. To this end, we begin with Figure 3 and add to it as shown in Figure 4; viz., we construct the square $AEDB$ on side AB, and in it draw segment DY parallel to BC, segments ES and BR

Figure 3



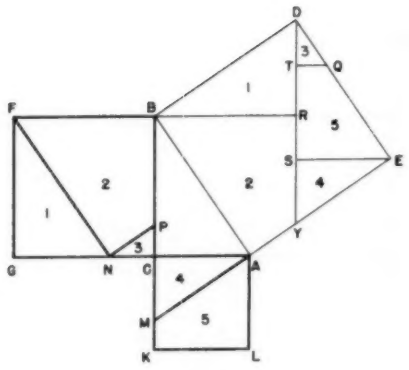


Figure 4

perpendicular to DY, and (after marking on DE the point Q such that $(DQ) = (NP)$)[2] the segment QT perpendicular to DY. Notice that AM is an extension of side AE, and that BR is an extension of side BF.

Now $(FN) = (AB)$, since FN and AB are segments cut from parallel lines by parallels. But $(AB) = (BD)$, hence $(FN) = (BD)$. Further, $\angle NFG = \angle DBR$, because these are acute angles whose arms are respectively perpendicular. We conclude that triangles FGN and BRD are congruent. Let us label each of these triangles "1."

Next, consider the two quadrilaterals $BFNP$ and $RBAY$. Since triangles FGN and BRD are congruent, we know that $(FG) = (BR) = (FB)$, in short $(BR) = (FB)$. In view of the fact noted earlier that $(FN) = (AB)$, these two quadrilaterals have two pairs of congruent sides and all corresponding angles the same; hence, by superposition, the two quadrilaterals are congruent. Let us label each of them "2."

Third, consider the two right triangles NCP and DTQ. By construction, $(DQ) = (NP)$. Also $\angle CNP = \angle TDQ$, because these two acute angles have arms that are respectively perpendicular. Hence, these two right triangles are congruent. We label each of them "3."

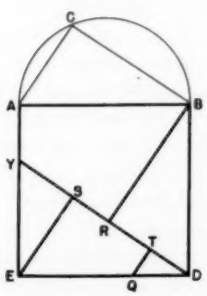
Fourth, let us turn to the right triangles ACM and ESY. These two triangles are similar, for their corresponding angles are equal. And, indeed, they are congruent since $(ES) = (AC)$. (This last follows from the congruence of the right triangles ACB and ESQ, which have corresponding angles equal and hypotenuses of the same length.) Label these two triangles "4."

Finally, consider the trapezoids $AMKL$ and $EQTS$. These trapezoids have two pairs of congruent sides: $(ES) = (AL)$ and $(EQ) = (AM)$ [the first, because $(ES) = (AC) = (AL)$; the second, because $(NP) = (AY)$ (by the congruence of quadrilaterals 2) and because $(EQ) = (DE) - (DQ) = (AE) = (AY) = (EY) = (AM)$]. Further, they all have corresponding angles equal. Thus, these two trapezoids are also congruent. We label them both "5."

Each pair of parts with the same numeral thus having been shown to be congruent, we have established that the five parts cut from our two original squares reassemble into a square.

A final remark. Our solution of Problem III makes clear how the converse of Problem III can be solved. Required to cut a given square $AEDB$ into parts which reassemble into two smaller squares, we construct a semicircle having side AB of the given square as diameter. Inscribing a right triangle ACB into this semicircle, we then proceed as Figure 4 suggests: draw DY parallel to BC, on ED locate Q such that $(EQ) = (EY)$, and then draw segments BR, QT and ES all perpendicular to DY. When $AEDB$ is cut along these

Figure 5



[2] We are continuing in force here a convention of our first paper to the effect that "(XY)" designates the length of segment XY.

four segments, the five parts which result can be assembled into two smaller squares. This *construction* is shown in Figure 5. The fact that the construction is effective may be *proved* by extending Figure 5 in the fashion of Figure 4. This last paragraph solves problem C posed at the end of our May 1956 paper in THE MATHEMATICS TEACHER. (Query: How can a square be cut so that its parts will reassemble into two congruent squares?)

PRELIMINARIES TO THE SOLUTION OF PROBLEMS IV, V AND VI

The solutions of these three problems use certain bits of information for school geometry which are best summarized at the outset.

An equilateral triangle whose side has length s has an area of $\sqrt{3}s^2/4$ and an altitude of length $\sqrt{3}s/2$. An equilateral triangle whose altitude has length h has an area of $\sqrt{3}h^2/3$ and a side of length

Figure 6

$2\sqrt{3}h/3$. These facts are obvious from the symmetry evident in Figure 6 and from the Pythagorean theorem.

In a different direction, recall that the mean proportional between two positive numbers q and r is the positive number p such that

$$\frac{q}{p}=\frac{p}{r}.$$

Such a number p therefore satisfies the equation $p^2=q\cdot r$, and so $p=\sqrt{q\cdot r}$. Clearly p is the mean proportional between q and r just in case p is the mean proportional between r and q.

In a right triangle, the altitude to the hypotenuse divides the triangle into two smaller triangles, each of which is similar to the original right triangle. Upon equating ratios of lengths of corresponding sides, various mean proportionals are disclosed. Thus, suppose as in Figure 7 that the sides of the right triangle have lengths d and b, its hypotenuse has length c, the

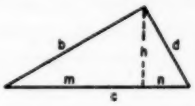


Figure 7

altitude to the hypotenuse has length h, and the foot of this altitude divides the hypotenuse into two segments of length m and n respectively; then

$$\frac{m}{h}=\frac{h}{n}, \quad \text{and so} \quad h^2=m\cdot n;$$

$$\frac{c}{d}=\frac{d}{n}, \quad \text{and so} \quad d^2=c\cdot n;$$

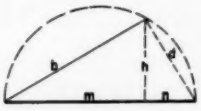
$$\frac{c}{b}=\frac{b}{m}, \quad \text{and so} \quad b^2=c\cdot m.$$

Noting that $c=m+n$, the last two equations above revise into $d^2=(m+n)\cdot n$ and $b^2=(m+n)\cdot m$.

Given two segments of length m and n respectively, we can conjoin them into a third segment of length $m+n$. Then the familiar construction shown in Figure 8 gives at one stroke the three segments whose lengths are respectively the mean proportionals h, d, b specified above, viz., $h=\sqrt{m\cdot n}$, $d=\sqrt{(m+n)\cdot n}$, $b=\sqrt{(m+n)\cdot m}$.

As we remarked earlier, these familiar facts will help us solve Problems IV, V, and VI. But it is worth noting that these facts are also useful in connection with the converse of Problem III. Thus e.g. suppose we are required to cut a given square $AEDB$ into parts that reassemble into two smaller squares of which one has an *area* k times that of the other. [We assume that $k\geq 1$. This assumption represents no loss of generality, since if one

Figure 8



square has an area k times that of the other, then the second has an area $1/k$ times that of the first.] Going back to Figure 5, we see that this problem can be solved simply by locating point C on the semicircle in such a way that $(BC)^2 = k(AC)^2$. Now notice from Figure 8 that $b^2 = kd^2$ if and only if $m = kn$, i.e. if and only if $a = n(k+1)$ or $n = a/k+1$. Thus we can find our point C by erecting a perpendicular from a point, say W, of AB which divides AB in the ratio $k:1$. Such a point W may be obtained by laying off on AB a segment BW whose length $(BW) = (AB)/k+1$. So long as k is a fraction (improper), segment BW is easy to construct. (When $k=1$, W is the midpoint of AB; this leads to the answer to our previous query.)

Solution of problem IV

In Problem IV we are asked to transform a given square into an equilateral triangle, i.e., by straight cuts so to divide the square into parts that upon rearrangement these parts form an equilateral triangle. Our solution of this problem again has two stages: a *construction* of the cutting lines in the given square, and a *proof* that the resulting parts reassemble into an equilateral triangle.

Label the vertices of the given square "A," "B," "C," "D," and let a be the length of any side of the square.

A necessary preliminary to our construction is the calculation of the length of the altitude of the desired equilateral triangle, and the geometric construction of a segment with this length.

The area of our given square is a^2. Hence the area of the equilateral triangle equivalent to this square must also be a^2. Suppose this equilateral triangle has a side of length x and an altitude of length h. Then $xh/2 = a^2$. Since it is necessarily the case for our triangle that $x/2 = h\sqrt{3}/3$, we have that $h^2\sqrt{3}/3 = a^2$, whence $h^2 = 3a^2/\sqrt{3} = a^2\sqrt{3}$ and

$$h = a\sqrt[4]{3}.$$

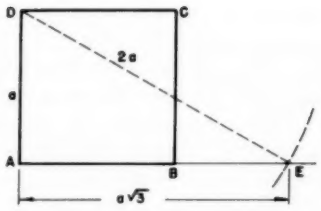


Figure 9

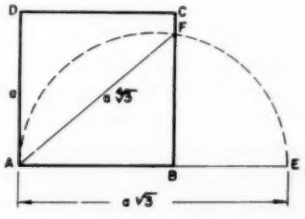
This result completes our calculation of the length h of the altitude of the desired equilateral triangle.

To construct a segment whose length is this number h, we use the fact that $h^2 = a\sqrt[2]{3} = a \cdot a\sqrt{3}$ and construct a segment whose length is the mean proportional between a and $a\sqrt{3}$. For this, we need a segment of length $a\sqrt{3}$. Such a segment may be constructed by the method illustrated in Figure 9; with vertex D as center, describe a circle whose radius has length $2a$; let E be the intersection of this circle with the extension of side AB; then the length of segment AE is $a\sqrt{3}$. That AE has length $a\sqrt{3}$ follows at once from the fact that DE has length $2a$ and the fact that DAE is a right triangle; for

$$(AE) = \sqrt{(DE)^2 - (AD)^2}$$
$$= \sqrt{4a^2 - a^2} = a\sqrt{3}.$$

With the help of ideas that center on Figure 8, we can use this segment AE and the side AB of the square to construct a segment whose length is the mean proportional between $a\sqrt{3}$ and a. With AE as diameter, construct a semicircle in the manner of Figure 10 and let F be the intersection of this semicircle with side BC.

Figure 10



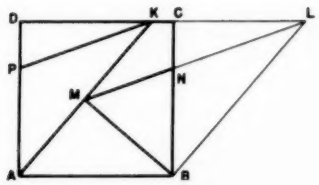


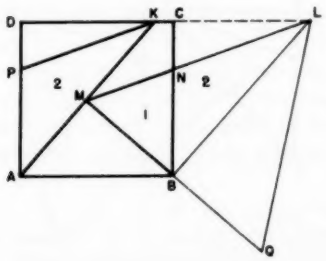
Figure 11

Draw the segment AF. The length of segment AF is the mean proportional between $a\sqrt{3}$ (the length of AE) and a (the length of AB). Thus AF has length $\sqrt{a \cdot a\sqrt{3}}$, i.e., $a\sqrt[4]{3}$, the length h of the altitude of the equilateral triangle we seek.

The *construction* of the cutting lines in our given square can now be described; the five successive steps of the construction are shown in Figure 11. (*a*) Returning to square $ABCD$, mark on side DC the point K such that segment AK has a length equal to that of the altitude of the desired equilateral triangle, viz., $a\sqrt[4]{3}$. (This can be done with the help of segment AF shown in Figure 10.) Draw segment AK. (*b*) From vertex B, draw the segment that parallels AK and that reaches DC extended at the point L. (*c*) From vertex B, also draw the segment that is perpendicular to BL and that ends on AK at the point M. (*d*) Draw segment ML, and let N be its intersection with side BC. (*e*) Finally, from K draw the segment that parallels ML and that ends on AD at the point P.

Now consult Figure 11. If our square $ABCD$ is cut along segment AK, and then along segments KP, BM and MN, the resulting five parts can be reassembled into an equilateral triangle. (This reassembling requires that some of the parts be turned over.)

Figure 12



The final stage of our solution consists in a *proof* that these five parts can be arranged into an equilateral triangle. To this end, we begin with Figure 11 and add to it as shown in Figure 12, viz., we extend segment MB to Q, where BQ has the same length as MB, and we draw segment QL.

Now MB and BQ have the same length, and BL is perpendicular to MQ; triangle MLQ is therefore isosceles, and ML and QL have the same length. We shall show that triangle MLQ is equilateral, and that it has the same area as square $ABCD$.

First, let us prove that *triangle MLQ is equilateral.* The construction of Figure 12 tells us that BL is parallel to AK, and that $(BL)=(AK)=a\sqrt[4]{3}$. Since MBL is a right triangle, we know that

$$(1) \qquad \begin{aligned}(ML)^2&=(BL)^2+(MB)^2\\&=a^2\sqrt{3}+(MB)^2.\end{aligned}$$

In view of the fact that the right triangles AMB and ADK are similar ($\angle DKA$ equals $\angle MAB$: these two angles are alternate interior angles cut from the parallels DC and AB by the transversal AK), it follows that

$$\frac{(MB)}{(AB)}=\frac{(AD)}{(AK)}, \text{ and so } \frac{(MB)}{a}=\frac{a}{a\sqrt[4]{3}};$$

hence

$$(MB)=\frac{a}{\sqrt[4]{3}}.$$

Substituting this value of (MB) in (1), we obtain

$$(ML)^2=a^2\sqrt{3}+\frac{a^2}{\sqrt{3}}=\frac{4a^2}{\sqrt{3}},$$

and so

$$(ML)=\frac{2a}{\sqrt[4]{3}}=2\left(\frac{a}{\sqrt[4]{3}}\right).$$

Thus $(ML)=2(MB)$. Since $(MB)=(BQ)$, we see that $(ML)=(MB)+(BQ)=(MQ)$.

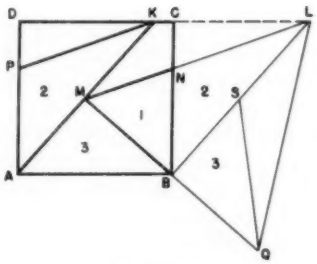


Figure 13

Therefore, $(ML)=(QL)=(MQ)$ and triangle MLQ is equilateral.

Finally, let us prove that *triangle* MLQ *is equivalent to square* ABCD, i.e., *triangle* MLQ *has the same area as square* ABCD.

Triangle BMN is common to both our square and our equilateral triangle; let us label it with the numeral "1" in Figure 12.

Again, triangle BNL and triangle APK are congruent, since they are similar (corresponding angles are equal) and since $(BL)=(AK)$. Let us label each of these two triangles with the numeral "2" in Figure 12.

Now we add to Figure 12, and obtain Figure 13, as follows: on segment BL mark the point S such that $(BS)=(MA)$, and draw the segment SQ. The right triangle SBQ resulting from this construction is congruent to the right triangle AMB, since MB is congruent to BQ and BS is congruent to MA. Let us label each of these triangles with the numeral "3" in Figure 13. (Note that triangle SBQ can be "superimposed" on triangle AMB only after triangle SBQ has been turned over.)

Figure 13 tells us that in our equilateral triangle MQL we have yet to account for the area of the triangle QSL; and that in our square $ABCD$ we have yet to account for the area of two parts—the right triangle PDK and the quadrilateral $MKCN$. Can triangle QSL be divided into two parts of such a sort that one is a triangle congruent to triangle PDK, and the other is a quadrilateral congruent to quadrilateral $MKCN$? The following considerations bear on this question.

Since triangles SBQ and AMB are congruent, we know that

(2) $$(QS)=(AB)$$

and that

(3) $$\angle BSQ=\angle BAM.$$

But $\angle BSQ$ is an exterior angle of triangle QSL; hence

(4) $$\angle SQL=\angle BSQ-\angle SLQ.$$

On the other hand,

$$\angle BAM=\angle AKD=\angle AKP+\angle PKD,$$

whence

$$\angle PKD=\angle BAM-\angle AKP;$$

but

$$\angle AKP=\angle MLB=\angle SLQ,$$

and therefore

(5) $$\angle PKD=\angle BAM-\angle SLQ.$$

Putting (3) and (5) together, we have

(6) $$\angle PKD=\angle BSQ-\angle SLQ;$$

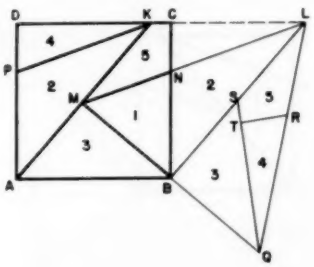
and finally, from (4) and (6),

(7) $$\angle SQL=\angle PKD.$$

The equality in (7) suggests the possibility of erecting a perpendicular segment opposite $\angle SQL$ and thereby obtaining from triangle QSL a right triangle congruent to triangle PDK. If we note further that $(QS)=(AB)=(DC)$ and $(DK)<(DC)$, it is clear that this perpendicular segment can be drawn from some point on QS.

Therefore, we add to Figure 13 and obtain Figure 14, as follows: on segment QS mark the point T such that $(QT)=(DK)$, and draw segment TR perpendicular to

Figure 14



QT. The right triangle QTR resulting from this construction is congruent to the right triangle KDP, since these two triangles are similar (with each angle of one there can be associated an equal angle in the other) and $(DK)=(QT)$. Let us label each of these triangles with the numeral "4" in Figure 14.

It is now a simple matter to show that the remaining part of the square and of the equilateral triangle, viz., the two quadrilaterals labelled "5" in Figure 14, are congruent.

Thus, we have *proved* that the equilateral triangle MLQ has the same area as square $ABCD$, and hence have completed our solution of Problem IV.

Notice that our solution is such that some of the parts cut from the square can be "superimposed" on corresponding parts of the equilateral triangle only if they are first turned over; this is true of triangle 3, triangle 4, and quadrilateral 5. It is an interesting question whether Problem IV has a different solution not requiring any part to be turned over. It is also interesting to seek a solution which sees the square cut into fewer than five parts.

Solution of problem V

In Problem V, we are asked to transform a given equilateral triangle into a square, i.e., by straight cuts so to divide the triangle into parts that upon rearrangement these parts form a square. Our solution of this problem has the same two stages that our solution of Problems III and IV had, viz., a *construction* of the cutting lines in the given triangle, and a *proof* that parts resulting therefrom can be reassembled into a square.

Since Problem V is the converse of Problem IV, it is reasonable to expect that the solution just completed of Problem IV can be utilized in our solution of Problem V.

Label the vertices of the given equilateral triangle M, Q, L, and let s be the length of any side of the triangle.

A necessary preliminary to our construction is the calculation of the length of the side of the desired square, and the geometric construction of a segment with this length.

Our equilateral triangle of side s has an area of

$$\frac{s^2\sqrt{3}}{4},$$

hence the square sought for must have this same area.

Let a be the length of the side of the desired square. Then the area a^2 of this square is such that

$$a^2=\frac{s^2\sqrt{3}}{4}.$$

Hence

$$a=\frac{s}{2}\sqrt[4]{3}.$$

Revising the right member of this equation thusly:

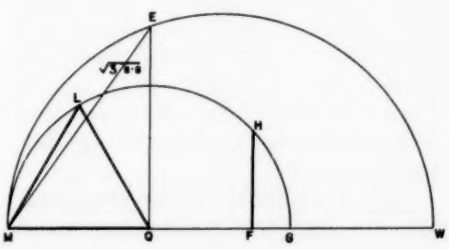
$$a=\frac{s}{2}\sqrt{\sqrt{3}}=\sqrt{\frac{s^2}{4}\sqrt{3}}=\sqrt{\frac{s}{4}\sqrt{3s^2}}$$

$$=\sqrt{\frac{s}{4}\sqrt{3s\cdot s}},$$

it appears that a segment of length a can be constructed as follows: First, construct the mean proportional between $3s$ and s; and next, construct the mean proportional between that result ($\sqrt{3s\cdot s}$) and $s/4$.

The construction is shown in Figure 15, where segment FH has the desired length $\sqrt{s/4\sqrt{3s\cdot s}}$. To obtain FH, we first locate W on the extension of MQ so that

Figure 15



$(MW) = 3(MQ) = 3s$, draw the semicircle with diameter MW, erect the perpendicular at Q, and draw segment ME whose length is $\sqrt{3s \cdot s}$. Then we mark F on the extension of MQ so that $(MF) = (ME)$, locate G there so that $(FG) = 1/4(MQ) = s/4$, draw the semicircle with diameter MG, erect the perpendicular at F, and draw the desired segment FH whose length $\sqrt{s/4\sqrt{3s \cdot s}}$ is the length a of the desired square

It would be a simple matter, now, to construct a square with side the segment FH of Figure 15. This square would have area equal to that of triangle MQL. Nevertheless, such a construction would *not* represent a solution of Problem V, since it furnishes no indication whatever of how to cut up our triangle and rearrange its parts into a square.

It is at this point that help comes from Figure 14 and the solution it embodies of Problem IV. We begin with triangle MQL and *construct* in it the lines shown in Figure 14; the successive steps of this construction appear in Figure 16.

In the equilateral triangle MQL, (*a*) draw the altitude LB and draw through vertex M a line parallel to LB; (*b*) on this parallel through M, mark a point A such that $(AB) = a$, where a is the length of the side of the desired square, and draw segment BA here we go back to Figure 15 for the segment FH and use this segment as a radius from B to strike an arc through the parallel; for FH was so constructed that $(FH) = \sqrt{s/4\sqrt{3s \cdot s}} = a$); (*c*) through vertex L, draw the line parallel to BA, and let K be the intersection of this line and the line through M; (*d*) from point B draw the segment BC perpendicular to BA, and let N be the intersection of BC and ML; finally (*e*) on BL mark the point S such that $(QS) = (BA)$, on LQ, mark R so that $(LR) = (MN)$, and then draw segment QS and segment RT perpendicular to QS.

Consult Figure 16. If our given equilateral triangle MQL is cut along segment

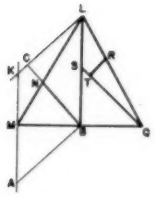


Figure 16

BL and then along segments BN, QS and RT, the resulting five parts can be reassembled into a square. (The reassembling requires that some of the parts be turned over.)

The *proof* that these five parts reassembled into a square runs as follows. First we add to Figure 16 and obtain Figure 17 thusly: draw segment AD perpendicular to BA.

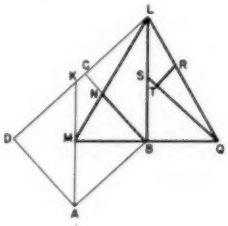


Figure 17

Now LD is parallel to BA, BC is parallel to AD, and $\angle ABC$ is a right angle. Therefore, the quadrilateral $ABCD$ is a rectangle. We must show that rectangle $ABCD$ is in fact a square, and that it has the same area as the equilateral triangle MQL originally given.

Let us show that *rectangle* ABCD *is a square*. The right triangles AMB and ADK are similar because $\angle MBA = \angle DAK$. (The arms of one of these acute angles being respectively perpendicular to the arms of the other.) Hence,

$$\frac{(BM)}{(BA)} = \frac{(AD)}{(AK)}$$

and so

$$(AD) = \frac{(BM) \cdot (AK)}{(BA)} .$$

But $(BM) = s/2$, $(BA) = a = s/2\sqrt{3}$, and

$(AK) = (LB)$. (This last because AK and LB are segments cut from parallels by parallels.) Since LB is the altitude of our given equilateral triangle, $(LB) = s\sqrt{3}/2$ and so $(AK) = s\sqrt{3}/2$. Therefore,

$$(AD) = \frac{\frac{s}{2} \cdot \frac{s\sqrt{3}}{2}}{\frac{s}{2} \cdot \sqrt{3}} = \frac{s}{2}\sqrt{3} = (BA),$$

and we see that rectangle $ABCD$ is a square.

Finally, we must show that *square* ABCD *is equivalent to our given equilateral triangle* MQL, i.e., *square* ABCD *has the same area as triangle* MQL. To this end we add to Figure 17, and obtain Figure 18, as follows: on AD mark P so that $(KP) = (QR)$, and draw segment KP.

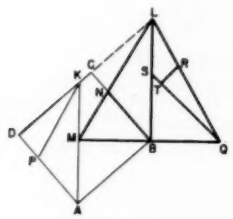


Figure 18

Taking into account all the steps of the construction shown in Figure 18, it is not hard to show that the five parts of square $ABCD$ have congruent matches in triangle MQL. The argument being entirely similar to that appearing in the solution of Problem IV, we do not repeat the details here.

Solution of problem VI

Problem VI requires us to transform a given non-rectangular parallelogram into a square, i.e., by straight cuts so to divide the parallelogram into parts that upon rearrangement these parts form a square.

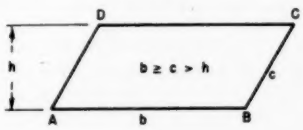
One approach to a solution of this problem consists in first transforming the given parallelogram into a rectangle (which is easily done with the help of one cut), and then inventing a partition of this rectangle into a square. Such an approach will, of course, succeed. However, it has the disadvantage of being *indirect*. The approach we take here is a *direct* one, viz., we shall show how the parallelogram itself can be transformed directly into a square.

We begin with a non-rectangular parallelogram. The lengths of two adjacent sides of this parallelogram are either the same or different, and in the latter event one length is greater than the other. Let us assume our parallelogram is so oriented that its base is *horizontal* and has a length *not less than* that of an adjacent side. Now let us label the vertices of the parallelogram A, B, C, D in counter-clockwise order and in such a way that segment AB is the base. Further, let the length of the base be b and the length of a side adjacent thereto be c; and finally, let the altitude of the parallelogram be h. The parallelogram of Figure 19 is drawn and labelled in accordance with these conventions. (It is to be emphasized that these conventions involve no loss of generality whatever: any parallelogram can be so oriented and so labelled.)

Thus, we are beginning with a parallelogram $ABCD$ such that $(AB) = (CD) = b$, $(BC) = (DA) = c$, and $b \geqq c$. Further, this parallelogram is non-rectangular, which means $c > h$. Because $c > h$, we know also that $b > h$.

Now the area of our parallelogram $ABCD$ is $b \cdot h$. A square equivalent to this parallelogram must therefore have a side of length $\sqrt{b \cdot h}$. Since $\sqrt{b \cdot h}$ is the mean proportional between b and h, it is a simple matter to construct a segment of length $\sqrt{b \cdot h}$ and, thereafter, to construct a square having this segment as a side. As we have seen in earlier Problems, how-

Figure 19



ever, such a procedure does not advance us towards a solution of Problem VI because it gives no indication of how our original parallelogram $ABCD$ is to be cut.

At this point, we establish two facts that steer us to the desired solution. The first fact is this: *Any non-rectangular parallelogram can be transformed into another such parallelogram with the same base and the same altitude, but with a second side that is the mean proportional between the base and the altitude of the original parallelogram.*

Consider a parallelogram $ABCD$ oriented and labelled as in Figure 19. Construct a segment of length a, $a = \sqrt{b \cdot h}$, by the mean proportional construction. With vertex A as center, draw an arc with radius a. This arc intersects side DC (or DC extended) *provided* a *is not less than* h; and since[3] clearly $h < a < b$, we see that our arc must cut DC. A point E of intersection of this arc with DC may therefore have one of the following three locations: (a) on the segment CD; (b) on the extension HD of CD; or (c) coincident with point D. The first of these cases is illustrated by the point E_1 in diagram (i) of Figure 20; the second, by E_2 in diagram (ii) of Figure 20; and the third, by E_3 in diagram (iii) thereof. In the first two cases, our given parallelogram can be transformed into a parallelogram having an altitude of the same length h and the same base segment AB, but with a second side AE, where $(AE) = \sqrt{b \cdot h}$. To this end (consult Figure 21), it suffices in case (i) to translate triangle $A_1D_1E_1$ to the position $B_1C_1F_1$, and in case (ii) to translate triangle $B_2F_2C_2$ to the position $A_2E_2D_2$. In case (iii), the given parallelogram already has a second side of length $\sqrt{b \cdot h}$, hence requires no transformation. These remarks establish our first fact.

The second useful fact about parallelograms is this: *Suppose a non-rectangular*

[3] Given that $a < h < b$, notice $h^2 < b \cdot h$ and $b \cdot h < b^2$, and so $h^2 < b \cdot h < b^2$. Consequently $\sqrt{h^2} < \sqrt{b \cdot h} < \sqrt{b^2}$, i.e., $h < a < b$.

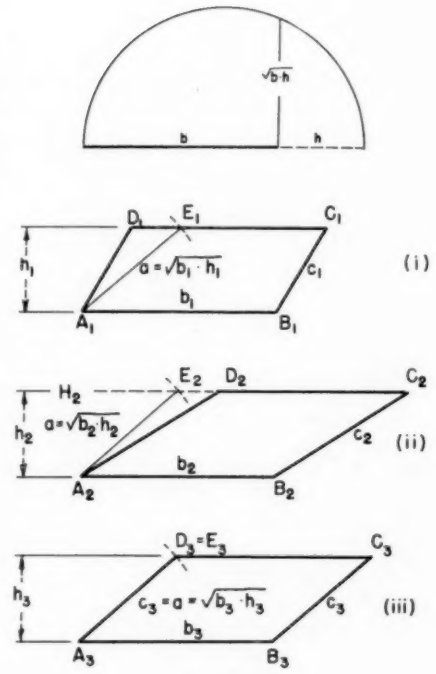
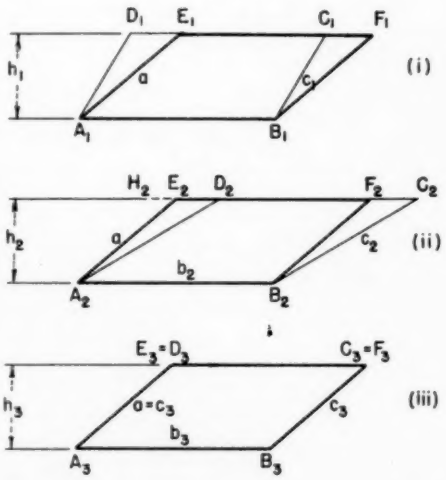


Figure 20

Figure 21



parallelogram ABFE *is such that the length* b *of its base segment* AB *is greater than the length* h *of its altitude, and further that its second side* AE *has the length* $\sqrt{b \cdot h}$; *then the segment* BK *perpendicular to the second side opposite vertex* B *also is of length* $\sqrt{b \cdot h}$.

To establish this fact, let us go back to parallelogram $A_1B_1F_1E_1$ shown in diagram (i) of Figure 21. (This parallelogram is such that $h<b$ and $(AE_1)=\sqrt{b \cdot h}$.) Now draw E_1L_1 perpendicular to A_1B_1, and B_1K_1 perpendicular to A_1E_1; the result is shown in Figure 22. Right triangles

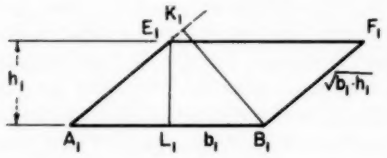


Figure 22

$A_1L_1E_1$ and $A_1K_1B_1$ are similar because they have $\angle K_1A_1B_1$ in common. From this similarity we have

$$\frac{(B_1K_1)}{(A_1B_1)}=\frac{(E_1L_1)}{(A_1E_1)},$$

and so

$$(B_1K_1)=\frac{(A_1B_1)(E_1L_1)}{(A_1E_1)}.$$

In view of the fact that

$$(A_1E_1)=\sqrt{(A_1B_1)\cdot(E_1L_1)},$$

it follows at once that

$$(B_1K_1)=\frac{(A_1B_1)(E_1L_1)}{\sqrt{(A_1B_1)\cdot(E_1L_1)}}$$
$$=\sqrt{(A_1B_1)\cdot(E_1L_1)}=(A_1E_1),$$

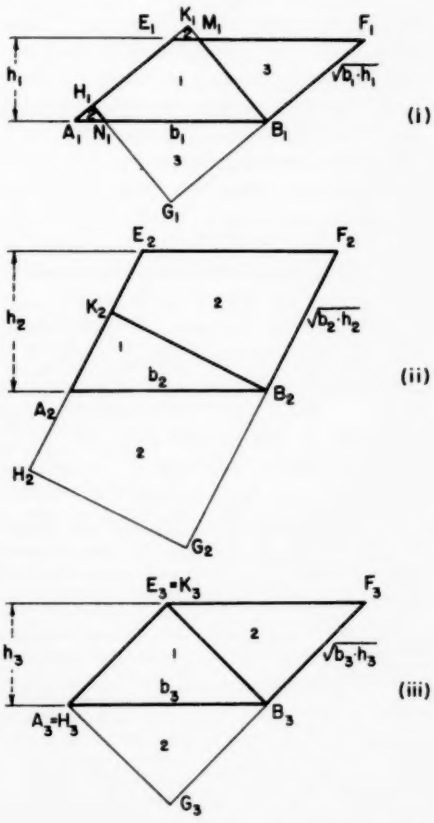
as was to be shown. (While this argument looks to a case in which the point K falls on side AE extended, the argument itself is applicable without change in the other cases—viz., when K falls on side AE, and when K coincides with E.)

Let us return to Problem VI and show first how it may be solved when the given non-rectangular parallelogram is a parallelogram $ABFE$ such that $(AB)=(FE)=b$, $b>h$, and $(AE)=(BF)=\sqrt{b \cdot h}$.

We proceed as follows: (*a*) draw the segment BK perpendicular to side AE (as we know, $(BK)=\sqrt{b \cdot h}$); (*b*) on side AE (extended, if necessary), mark the point H such that $(KH)=(BK)=\sqrt{b \cdot h}$; (*c*) on side BF extended, mark the point G such that $(BG)=(BK)=\sqrt{b \cdot h}$; and finally (*d*) draw the segment HG. The three diagrams of Figure 23 illustrate this procedure for each of the three possible locations of point K.

The procedure just described amounts

Figure 23



to a *construction* of the cutting lines in our parallelogram $ABFE$. Thus, if parallelograms of the type shown in diagram (i) of Figure 23 (viz., the type in which K falls on the extension of AE) are cut along BM and NH, the resulting three parts may be assembled into a square. Similar remarks apply to the types shown in diagrams (ii) and (iii), where only one cut—along BK—is necessary.

To *prove* that this construction yields a square in each case, we must show that the quadrilateral $BKHG$ is always a square, and that parts labelled with the same numeral are congruent. These are simple matters which we omit.

Hence, finally, we see how *any* nonrectangular parallelogram may be cut up and assembled into a square. The parallelogram is first transformed into one whose second side has a length that is the mean proportional between the lengths of base and altitude respectively. The resulting parallelogram is then divided in the manner just described. No more than three cuts and four parts are required for any parallelogram.

Envoi

We conclude with two questions that can be answered with the methods of the present paper:

A. Devise a construction by which a square is transformed into two congruent equilateral triangles.

B. Devise a construction by which a square is transformed into three congruent equilateral triangles.

Here is a hint on Question A: if two congruent equilateral triangles are put together appropriately, they form a parallelogram whose height is the height of the triangle and whose base is the base of the triangle. The corresponding hint for Question B runs as follows: if three congruent equilateral triangles are put together appropriately, they form a trapezoid whose height is that of the triangles and whose longer parallel side has a length twice that of the side of the triangles; and by one cut (through the midpoints of two sides of the right-hand triangle) this trapezoid can be transformed into a parallelogram whose base has a length one and one-half times the length of a side of the triangles.

"If science deals with quantitative problems of a purely logical character, if science has no recognition of, or concern for, value or purpose, how can modern scientific man achieve a balanced good life, in which logic is the companion of beauty, and efficiency is the partner of virtue?

"In one sense the answer is very simple: Our morals must catch up with our machinery. To state the necessity, however, is not to achieve it. The great gap, which lies so forebodingly between our power and our capacity to use power wisely, can only be bridged by a vast combination of efforts. Knowledge of individual and group behavior must be improved. Communication must be improved between peoples of different languages and cultures, as well as between all the varied interests which use the same language, but often with such dangerously differing connotations. A revolutionary advance must be made in our understanding of economic and political factors. Willingness to sacrifice selfish short-term interests, either personal or national, in order to bring about long-term improvement for all must be developed."—*Taken from* Science and Complexity, *by Warren Weaver,* American Scientist, *Vol. 36, No. 4*

Honors mathematics at Dartmouth

JOHN G. KEMENY, *Dartmouth College, Hanover, New Hampshire.*

To find a way to unshackle the gifted is a major problem in college as well as in the high school.

IN NO OTHER FIELD is there as great a difference between the average and the outstanding student as there is in mathematics. Having the two groups in one class will work to the disadvantage of both groups. The exceptionally able man will ask questions way beyond the average student, while the latter will bore the former to tears.

Yet in an age when there is a great shortage of mathematically trained men, both groups must be encouraged to take college mathematics courses. The only reasonable solution is to have different programs for the two groups. This article deals with a special program designed for the exceptional students. The program is now in its third year at Dartmouth College, and it has proved to be very successful.

College mathematics courses fall naturally into two groups: those of the first two years are taken by all students preparing for the physical sciences or for engineering, while those in the last two years are taken mostly by men with a special interest in mathematics for its own sake. The former courses are all basic calculus courses, while the latter introduce the student to a variety of advanced mathematical fields. An adequate honors program must consist, correspondingly of two basically different parts.

THE FIRST TWO YEARS

Freshmen often complain that they expected college to be something new and exciting, and yet college mathematics is pretty much like high school, only the subject-matter is different. If we are to avoid losing the most able students through boredom, they must be separated as soon as they arrive in college. And that means that they must be selected *before* they arrive.

At Dartmouth we examine the aptitude scores and recommendations of all entering freshmen during the summer before they arrive. The top 10 per cent of the class is selected from the point of view of mathematical promise. They are asked by mail whether they would be interested in going into an honors section of the freshman calculus course. It is carefully pointed out that this will involve extra work on their part. But we can assure them that they will not be penalized in their grades. Most of the students accept the offer.

When the students actually arrive, a final selection is made. Originally the number was restricted to 40, but as aptitude scores of freshmen improved considerably over the past few years, the number was increased to 60.

The first semester of the calculus sequence at Dartmouth introduces the student to the basic ideas of analytic geometry, and both the differential and integral calculus. This is accomplished by restricting work to polynomials. The first semester of the honors sections covers essentially the same material. But much less time is devoted to drilling the student and more to an understanding of fundamentals. In addition, some topics are added from time to time, e.g., mathematical induction. But throughout the first semester we keep in mind that our method of selection is not perfect; it must

be possible for students to transfer to or from an honors section.

At the end of the first semester a number of students decide to leave (or are asked to leave) the honors sections. And a very few students are allowed to transfer from a regular section to an honors section. This gives us a truly exceptional group.

In the second semester the sections switch to a new text, Courant's *Differential and Integral Calculus*. This is a book that combines absolute mathematical rigor with an attempt to develop the student's intuition. It is also very strong on showing how to bridge the gap between theory and application. But it is much too difficult for the average undergraduate. The book provides a real challenge for outstanding students, and all of them feel that it is well worth the necessary extra effort.

During the next three semesters every effort is made to keep the sections intact and to have the same professor take them through Courant. During these three semesters the teacher is able to get to know his students personally and to find out all their strengths and weaknesses. The sections also develop a strong *esprit de corps* which encourages a maximum effort. For example, although attendance is not taken, it is nearly perfect.

The emphasis throughout the course is on a thorough understanding of basic ideas. Technique is developed primarily through homework exercises. The students cover practically all of volume one of Courant and most of volume two. This is the equivalent of the traditional two-year sequence, plus advanced calculus. But the total amount covered in itself does not give a full picture of the effect of honors sections. After two years even the weakest man in the honors section has a much better understanding of the nature of mathematical reasoning than the best man in a regular section. Problems that we wouldn't dare give to a regular section turn out to be too easy in honors sections.

An important feature of the program is that the students acquire the habit of asking questions. They will not let anything pass that hasn't been fully explained. And they confront their instructor with a wide assortment of questions about mathematics. Often the entire class-period is devoted to some totally irrelevant topic. But through this process the students develop a love for mathematics.

The last two years

About half the students who start in an honors section stay with the program for the full two years. Those who finish go into many different fields. While most of them go into mathematics, physics, chemistry, or engineering, it is not uncommon to find a future doctor, lawyer, or businessman amongst them. Many of these continue with more advanced mathematics courses, to fulfill the curiosity aroused during the initial years.

A few of the very best students decide to major in mathematics. These are designated as "honors majors," and receive professional training especially designed for them.

Roughly speaking, the honors majors are able to omit the courses usually taken in the junior year. They go directly into advanced courses designed to introduce them to modern mathematics. There are year courses in geometry, probability and statistics, real and complex variables, abstract algebra, mathematical methods of physics, and logic and foundations. The second semester of these courses is frequently of graduate caliber.

In addition to these, there are occasional seminars arranged especially for the honors majors, when they request one. And they are all expected to write a thesis in their senior year, based on some independent work under the supervision of an adviser.

An honors major manages to take most of the advanced courses, plus some special work. By the time he graduates, he has

enough credits in mathematics to qualify for an M.A. at most schools. We expect that nearly all our honors majors will go on to a Ph.D. in mathematics. Our experience so far leads us to hope that we will send on three or four men each year to advanced graduate work, with excellent preliminary training. Since in recent years the U.S. has averaged only 200 Ph.D.'s in mathematics per year, this would be a major contribution for a liberal arts college. It would be more than ten times as much as the college was able to do in the past. The major credit for this phenomenal increase should go to our new honors program.

But, as a teacher, I must add a final note. No matter how happy we may be to send on so many able men to advanced work in mathematics, I must confess that we have a purely selfish motive which in itself justifies the effort that goes into the program. Every man in our department who has been fortunate enough to teach an honors section of calculus or one of the special advanced courses has felt that this was one of the most stimulating and rewarding experiences in his teaching career

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Have you read?

"The Age of Research," *Time*, July 9, 1956, pp. 74–76.

This report of America's five billion dollar investment for abundance is an excellent guidance article for your mathematics students. It not only points out the present and future benefits of scientific research but it gives some important insights into how these gains are made.

For example in 1946 there were two nonprofit research centers; today there are forty-eight. Research is now a cooperative effort. For example, the Chrysler Corporation has 200 scientists and mathematicians working together. Physicists, chemists, mathematicians, atomic scientists, neurologists and many others are all working together. Vast sums of money are spent; for example, Bell Telephone spends over one hundred million dollars for their research chemists. Here is an article that will amaze your students as to the magnitude and potential of research.

Conly, Robert Leslie. "Men Who Measure The Earth." *The National Geographic Magazine*, March, 1956, pp. 335–362.

No mathematics teacher should fail to recommend this article as reading material in his classes. Any American school boy will take a second look at mathematics after reading it.

It is a word and picture story of the activities of the Inter-American Geodetic Survey. The purpose of this survey is to accurately map 18 of the new world countries. It is a cooperative venture between North and South America. It is interesting to note that much of the work is done at night on mountains 15,000 feet high; that some of the instruments are so delicate that they measure to within one ten-millionth the effect of the earth's crust on the pull of gravity. Your students will be surprised to learn that much of the work is based on measurements from the stars and that the time signals come directly from the Washington, D. C. Bureau of Measurements. Problems in triangulation, leveling, aerial photography, and the use of the astrolabe are all discussed.

Be sure not to miss this article for a very comprehensive story of the geodetic survey and how, as a result of it, we will be able to locate precisely many things and places we have not known existed. It might help to have your director of guidance also read it.

Fry, Thornton C. "Mathematics as a Profession Today in Industry." *American Mathematical Monthly*, February, 1956, pp. 71–80.

As mathematics teachers we are always on the alert for information to help our students see the place of mathematics in the world about us. Every guidance person and mathematics teacher should read this article. It shows how the contribution that mathematics makes to industry has changed in the past few years. For example, in 1900 Steinmetz worked with General Electric and Campbell with Bell Telephone. These two almost covered the field. In 1940 there were only about 150 mathematicians in industry, in 1955 there were approximately 1500. What will happen in the future? Remember, the students taking mathematics today are preparing for the next 40 years. Think of all the new inventions since 1900; TV, radio, cellophane, guided missiles, and so forth. Think of the increased precision and the nature of problem solving. It is now a group project; mathematicians are members of a team. What does this all mean? Essentials of mathematics must be compressed into fewer years so greater progress may be made. But read the article. It will give you help.—Philip Peak, *Indiana University, Bloomington, Indiana*

The national status of mathematics contests

DANIEL B. LLOYD, *District of Columbia Teachers College, Washington, D.C.*

Ways to stimulate interest in mathematics are being sought by all. While mathematical contests and examinations are not universally looked upon with favor, their use is increasing.

MATHEMATICS CONTESTS have made a worthwhile contribution toward stimulating student interest in mathematics in this country for many years. A gradual upswing of interest in contests has been noticeable in the last decade since World War II.

Healthy competition in all fields of endeavor has been the chief characteristic of the free-enterprise system since the founding of our republic. It has been America's secret weapon in attaining supremacy. With reference to the field of dramatic expression, it was Shakespeare who voiced the sentiment, "The play's the thing!" For the purpose of stimulating superior effort and achievement in practically any field, we can well paraphrase this to say, "The contest's the thing!"

Although adopted successfully in this country, the mathematics contest is not original with us. Such contests have been operating on a nation-wide scale in Hungary for over 60 years, in Russia for 30 years, and in other countries almost as long. Undoubtedly many mathematicians received their first impetus from such contests. A case in point is Hungary, which has produced as high a ratio of mathematicians per 100,000 population as any country in the world.

The William Lowell Putnam Mathematical Competition,[13] for undergraduates in colleges in the United States and Canada, is sponsored by the Mathematical Association of America—the director being at Kent College in Ohio. The association has just completed its sixteenth annual competition. For six hours of written examination, numerous cash prizes, up to $2000, are awarded the winners. A number of leading young American mathematicians first won their spurs in this competition; among these are Arens at UCLA, Gleason at Harvard, and Kaplansky at Chicago.

Pi Mu Epsilon, honorary mathematics fraternity, through its various chapters, annually awards numerous prizes and scholarships to college students, on the basis of competitive examinations.[6]

In addition to these college contests, it is estimated that some 60 or more mathematics contests for high school students are held each year in this country. In such cities as Chicago, New York, and Los Angeles, city-wide contests are conducted by local colleges and universities. There are also several well-known statewide contests, such as: (1) Indiana University High School Achievement Program, operating for 26 years; (2) the Stanford University Competition,[11] now running through its twelfth year, covering the four states of California, Arizona, Oregon, and Washington; (3) the Metropolitan New York Contest,[5] which has been widely operating for eight years, not only throughout New York City and New York State, but also in Colorado, Illinois, Nebraska, Oregon, Washington, Wisconsin, and other states, in all of

which the same examination, constructed by the Metropolitan New York committee, is administered to nearly 25,000 contestants; and (4) the Maryland-District of Columbia-Virginia Section of the M.A.A., starting its fourth successful season with the high schools of those states.

Indiana and Louisiana are typical of state universities which hold a state-wide interscholastic meet, in which mathematics is only one of several academic subjects tested. The main purpose is to attract able students for college scholarship offers. A number of private colleges have a similar plan.[7]

Another significant practice emphasizing mathematics is an increasing trend to include mathematics among the several basic sciences when science fairs and competitions are held. This is true in the Westinghouse Talent Search, with its associated local science fairs; in the New England School Science Fair; and in others.[7]

Mayor, in 1949, surveyed the scholarships and awards programs existing among the colleges.[7] The U.S. Office of Education, every few years, publishes a comprehensive list of available college scholarships.[10]

Current high school contests

Early in 1956, the author made a survey of mathematics contests held for high school students. For that purpose a questionnaire of 20 questions was distributed to all 48 states and the District of Columbia through the state representatives of the National Council, and also to 80 other leaders in the field of mathematics education.

The replies yielded 43 contests, scattered throughout 29 states, involving some 4200 schools, including a few in Canada. Table 1 on p. 460 lists them by states and shows the number of schools competing, form of examination given, and sponsoring agency.

The entrance fee paid by schools ranged from \$10 to nothing (the mode). Others charged from 10 cents to \$1 per student. Funds needed, in excess of fees collected, were contributed by the sponsoring organization, or in rare cases through gifts from anonymous businessmen or firms.

The form of written examinations is usually objective, short-answer questions to facilitate scoring, unless the sponsor, such as a college, is interested in testing organized thinking, such as problem-solving ability. The Stanford examination is of this broad-problem type. (Professor Polya has been constructing these tests since his retirement.)

In most cases the scoring of papers is done by high school teachers in the participating schools, unless machine-scoring is used. College-department members score many, and in some cases use superior students, math club members, and undergraduate math majors.

Some distinctive types

1. The "tournament" type of contest has proven popular in some places. Typical of this is where each school sends a team of four members to the college campus on a Saturday morning in May. One member from each team competes in a test of speed and accuracy, answering questions flashed on a screen. Next, everyone submits to an hour's written test. In the third round, each school competes as a team in a group test.

2. The Ohio program is a state-wide scholarship test covering a range of academic subjects. Students in the upper quartile of their classes may enter this preliminary test, a screening test. The survivors are eligible for the final test, in which only one subject may be chosen. This is an hour-long written test which is machine-scored.

3. The Texas Interscholastic League is sponsored by the extension branch of the University of Texas. The number sense contest, which consists of two divisions, one for elementary and one for high schools, is a test in mental arithmetic.

TABLE 1

State*	Number of Schools: Public	Private	Total†	Form of Exam‡	Percentage of Eligibles Competing§	Sponsor
Alabama	23	3	26	W,C,T	33	Howard Coll., Birmingham
Arizona	34	5	39	W	60	Ariz. Assn. of T. of Math.
California			82	W		Stanford Univ.
California	35	5	40	W,T	90	Los Angeles City Coll.
California			20	W		Pasadena City Coll., Pasadena
Colorado	24	1	25	M.A.A.	15	Colo. Educ. Assn., E. Div.
D.C. (& Md. and Va.)	66	18	84	W	40	Md.-D.C.-Va. Section, M.A.A.
Illinois			40	W		Knox Coll. (scholarship)
Illinois	16	10	26	W		Wright Jr. Coll., Chicago
Illinois		30	30	T,W	50	Mundelein Coll., Chicago
Illinois			88	M.A.A.	12	Ill. section, M.A.A.
Indiana	365	21	386	T,W	50	Ind. Univ., Bloomington
Louisiana			125	W	60	La. State Univ.
Maryland (see D.C.)						
Massachusetts	15		15	T	15	Newton H.S., Newton
Massachusetts	60	4	64	W	15	Univ. of Mass. Math. Club
Montana	100	3	103	W	60	Mont. State Coll., Bogeman
Montana			180	W	25	Pi Mu Epsilon, State Univ.
Nebraska			30	M.A.A.	20	Neb. section, M.A.A.
Nebraska	20		20	W		Neb. State Teach. Coll., Kearney
New Jersey	85	12	97	W	50	Rutgers Univ. & Assn. of Math. Teach. of N.J.
New Mexico	55	4	59	W	60	N.M. Educ. Assn., Math. section
New York			20	W	80	Coll. of St. Rose, Albany
New York	50	10	60	W		Manlius School, Manlius
New York	15		15	W	30	Interscholastic League, Lincoln H.S., Brooklyn
New York			183	M.A.A.	25	Metro. N.Y. section, M.A.A.
New York	105	35	140	W	25	Wash. Sq. College, N.Y.U.
New York	33	13	46	M.A.A.	40	Buffalo Bd. of Educ. & Upper N.Y. section, M.A.A.
North Carolina			64	W		Univ. of N.C.
Ohio			757	W	75	State Dept. of Educ., Columbus
Ohio			22	W		Greater Toledo Council of Teach. of Math.
Oklahoma	75	2	77	W	60	Central State Coll., Edmond
Oklahoma			50	W		Univ. of Okla.
Oregon	62	1	63	M.A.A.	33	Univ. of Ore., Eugene
Pennsylvania	200	50	250	T		Pa. Council of Teach. of Math.
South Dakota	7	1	8	W	50	Black Hills Math. Teach.
Tennessee	26	3	29	W	33	Austin Peay State Coll.
Tennessee	13		13	W	80	Lincoln Memorial Univ.
Tennessee			100	W		E. Tenn. Educ. Assn.
Texas	600		600	S	60	Univ. of Texas, Austin
Virginia (see D.C.)						
Virginia	86		86	T		Va. Conf. of Sc. & Math. Teach., Hampton Institute
Wisconsin	7	13	20	W	42	Mt. Mary Coll., Milwaukee
Wisconsin (M.A.A. contest starting in 1956)						
Washington			75	M.A.A.	25	Pacific N.W. section, M.A.A.
Wyoming (contest starting in 1956)						

For footnotes, see facing page→

The slide rule contest consists of 60 problems of such difficulty that no one completes them in the 30 minutes allotted. District winners qualify for regional meets and regional winners for the state meet.

4. The Indiana High School Achievement Program started in 1913; the mathematics was included beginning in 1930. The regional preliminary tests are given at 35 centers; then the finalists meet at Bloomington. Besides the tests, entertainment, exhibits, and sports are provided to create a relaxed and festive atmosphere.

5. The Interscholastic League, of New York City (present headquarters in Brooklyn), started in 1920 with an algebra test; it later expanded to include all high school mathematics. Five contests are conducted each semester, ten a year. The selection of contestants, and their coaching within the schools, vary considerably. Those so chosen become an accelerated group within the school and an additional burden on the teacher. All teachers are not sympathetic to the program.

6. Lincoln Memorial University in Tennessee invites students from the tri-state-area high schools to compete in a one-hour test in algebra 1, or algebra 2, and/or plane geometry. There are team honors as well as individual awards. The afternoon features a tour, ball game, and swimming.

Nation-wide contest—pro and con

In this nation-wide survey of contests, the last question asked on the questionnaire was:

20. Please give the views of those in charge of the contest, concerning the following questions:
A. Would you favor a national contest, sponsored, e.g., by the NCTM or the MAA, with a single exam, prepared by a central committee, and mailed to the various local cooperating organizations, such as yours; with local, state, and national winners (or some variation thereof)? Comment.
Or B:
Would you prefer the national body to furnish merely advisory service to local groups, such as sample questions, suggestive plans and patterns for contests; as in a Contest Handbook, for optional use only? Comment.

Gleaned from the replies were found the following potent arguments *against* a nationally sponsored contest:

1. Differences among high schools in mathematical training offered. Thus, Seattle says: "We are not interested in a national contest, for our students do not take the traditional high school courses. We know from years of experience that our students from Mathanalysis (an integrated course in trig, college algebra, and analytic geometry, with some calculus) are better prepared than other students for college work."

2. Unfair competition with stronger schools in larger urban areas against weaker schools in less populated areas.

3. Numerous colleges give their own type of tests, with emphasis on aptitude, for awarding scholarships.

4. Possible abuses by some teachers caused by the pressure of a big national contest. This would encourage boning and coaching for the contestants at the expense of the others and the neglect of the course of study.

5. The burden on the sponsoring organization in administering such a gigantic program.

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* States planning to start contests in 1957 are: Minnesota; West Tennessee; and Suffolk County, N.Y. Arkansas discontinued its contest in 1940. The following states reported no contests: Connecticut, Florida, Georgia, Kansas, Maine, Michigan, Mississippi, Nevada, New Hampshire, Rhode Island, Vermont, West Virginia. The following states failed to report: Delaware, Idaho, Iowa, Kentucky, Missouri, North Dakota, South Carolina, Utah.

† Total schools, all states: 4200 (approx.).

‡ Key to form of examination:
C—Ciphering matches.
S—Slide rule test and number sense test.
T—Team contest, or tournament.
W—Individual written examination, assembled.
M.A.A.—Written examination, furnished by the N.Y. Metropolitan Section of M.A.A.

§ Average of eligible schools: 40 per cent (approx.).

6. Opposition of some school officials to staging contests of any kind.

7. More interesting to have local participation of teachers and others in preparing and administering a contest.

8. A standard uniform examination would not satisfy many states which have their special pattern of contest already established by tradition or preference; e.g., team contests and tournaments.

9. "If the purpose of a contest is to increase interest in the study of mathematics, then various alternatives to a contest must be considered. Other activities may be more effective in increasing interest in mathematics; for example, a science and mathematics day was held on this campus last Saturday which brought 1000 students and teachers to hear special lectures, to see exhibits and to discuss projects for high school students. Members here feel that activities of this sort are far more effective in increasing interest than contests. On the other hand, if the purpose of the contest is merely to locate those with talent, then the effort need not be organized on a national scale but can be left to local states and communities," points out G. B. Price, University of Kansas.

10. "If testing is to identify the talented in mathematics, we should devise an instrument to measure the essential attributes. I regard these to be creative ability, imagination, originality, and a capacity to use these qualities. A test to measure these should involve oral, written and concrete products. Thus a written test, an exhibit, and an oral demonstration should be included. From this a student will demonstrate not only originality and aptitude, but also resourcefulness in carrying through a project," suggests D. A. Johnson, University of Minnesota.

The following stronger views were also received to the question: "Do you favor a national contest?"

This reply came back: "NO! No central testing agency has any right to exist in Democratic Society. A "contest" places too much false value on learning and education. As for tests and their use, please let local and state groups determine their own programs."

Newton, Massachusetts: "We do not consider an examination a contest. Our contest is a tournament on an elimination basis and is quite exciting."

The predominant sentiment in the following areas favored option B, i.e., limiting any national assistance to advisory aid or other minor aid: Washington (Seattle), California, Texas, Minnesota, Wisconsin, Ohio, Middle Atlantic States, and New England.

From the Stanford committee comes this interesting statement: "A national contest as an ultimate goal seems to us very desirable. Yet to introduce it at once seems to be difficult. With our type of examination, which we prefer, the main difficulty would be uniform scoring. [They use a smaller number of broad problems, similar to the Putnam Examination on the college level.] The right plan may be to attain the national examination by stages; start with local groups, encompassing a few states (as our examination), then enlarge the groups, and attain the national contest as a final goal. In the first stages, different types of examinations may be tried, in the hope that sufficient experience will lead to the most suitable type."

In some 15 states there was considerable sentiment expressed favoring a national contest, but with caution concerning local and state needs, as to awards, scholarships, etc. This sentiment is stronger in the less populated states—the southwest and northern plains, a few southern states, and scattered parts of eastern states. Thus one mathematics group, voting overwhelmingly for it, said: "This could become a very outstanding contribution by the NCTM to the growing demand for encouragement to the more capable students in mathematics."

Dr. C. L. Thiele, a director of exact sciences in Detroit, said: "I feel we are in need of a national contest such as the one you describe. I would favor a contest spon-

sored by the NCTM, with a single examination."

At the 1955 Christmas meeting of the NCTM in Washington, D.C., an enthusiastic section meeting voted overwhelmingly for the principle of a national contest. Sentiment was even heard favoring an *international* contest.

A nation-wide contest would evoke widespread interest but would involve numerous problems. It would entail enormous labor and initial financial support. It should not interfere with local contests already operating successfully. The need appears greatest in virgin territory, identified by this survey, where no contests have been reported. Thus this survey reveals a challenging opportunity to develop a wider program—if not to launch a nation-wide contest, at least to strengthen and increase contests on the local level.

The National Council has consistently declined to take the initiative in launching a nation-wide contest, but would cooperate with an MAA contest, basing its argument on three main points: (1) the MAA membership is more detached from the competing schools and therefore can handle the contest more impartially; (2) joint or divided administration would be more cumbersome and less efficient; and (3) persons with valuable contest experience, who might be most helpful, are members of both organizations and already are active in contest work.

Our survey of contests has been helpful in shaping the policy of the MAA. The latter has recently decided to plan for a nation-wide contest and is now canvassing the sentiment of its members concerning it. It seems possible that such a project may get started in a couple of years. It could well emerge as an outgrowth of their present sectional contests, particularly the Metropolitan New York one, by far the largest one in the country. It would have great potential in attracting support from wealthy foundations and corporations. We wish them the greatest success in this undertaking.

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The author wishes to acknowledge with appreciation the assistance of Miss Frances C. Johnson, of Oneonta, New York, who aided in analyzing the survey material.

Are your standards in marking changing?

R. F. GRAESSER, *University of Arizona, Tucson, Arizona.*

Have you ever studied the "drift" in your grading? If interested, the design for a study of your grades can be found in this article.

ARE YOU BECOMING more strict or more lenient in the semester marks which you give to your students? If you have kept a record of your distributions of marks over a number of semesters, you can easily answer this question. There are several statistical approaches to the answer. Of these, the theory of runs is perhaps the easiest to understand. Its development requires only a little algebra and no knowledge of statistics. To proceed to that theory, if we have seven a's and seven b's arranged, say, $a\ a\ b\ b\ a\ a\ a\ b\ b\ b\ b\ b\ a\ a$ (which may be written $a^2b^2a^3b^5a^2$), we are said to have three runs of a and two runs of b. A run is a succession of identical letters which is preceded and followed by another letter or by no letter at all. Is the above arrangement of letters a random one due to chance, or has it been influenced by some external cause?

Light may be thrown on this question by considering the probability of obtaining three runs of a's and two runs of b's by chance alone, assuming that all of the possible permutations of the seven a's and seven b's are equally likely. Let the probability ratio for this event be f/t. Then f is the number of permutations of seven a's and seven b's which will have three runs of a's and two runs of b's. Consider first the a's. In the permutation $a^2b^2a^3b^5a^2$ there must be three factors that are powers of a, since there are three runs of a. The product of the three factors must be a^7, since all seven a's must be used. We could have $a^2a^3a^2$, or $a^4a^2a^1$, or $a^2a^2a^3$, etc., each representing three runs of a. The exponents must be chosen from the numbers 1, 2, 3, 4, 5, 6, 7 in such a way that their sum is seven. The question is in how many ways may this be done? To answer the question consider the product

$$(1)\qquad (a+a^2+\cdots+a^7)(a+a^2+\cdots+a^7)\cdot(a+a^2+\cdots+a^7).$$

To obtain the expansion of this product, we must multiply together a term chosen from each parenthesis in every possible way and add the results. To obtain the term of the expansion with a^7, we must choose a term from each parenthesis in every possible way so that the sum of their exponents shall be seven. The number of ways in which this may be done is the coefficient of a^7 in the expansion. It is also the answer to our question. To obtain this coefficient, (1) may be written as

$$\begin{aligned}(a+a^2+\cdots+a^7)^3 &= a^3(1+a+\cdots+a^6)^3 = a^3\left(\frac{1-a^7}{1-a}\right)^3\\ &= a^3(1-a^7)^3(1-a)^{-3}\\ &= a^3(1-3a^7+3a^{14}-a^{21})\\ &\quad\cdot(1+3a+6a^2+10a^3+15a^4+\cdots).\end{aligned}$$

Hence the coefficient of a^7 is 15, which is also the number of permutations of seven a's that will contain three runs of a.

To find the number of permutations of seven b's that will contain two runs of b's,

we proceed in the same way. This time we need the coefficient of b^7 in the expansion of

$$(b+b^2+\cdots+b^7)^2$$
$$=b^2(1+b+\cdots+b^6)^2=b^2(1-b^7)^2(1-b)^{-2}$$
$$=b^2(1-2b^7+b^{14})$$
$$\cdot(1+2b+3b^2+4b^3+5b^4+6b^5+\cdots).$$

Hence the coefficient of b^7 is six. But the three runs of a can be combined with the two runs of b in 15 times 6 ways. Hence f is equal to 90. Next, t is the number of permutations of seven a's and seven b's, i.e., it is the number of permutations of 14 things, seven alike of one kind and seven alike of another kind. The number of these permutations is $14!/7!7!=3432$. Finally, the probability of obtaining three runs of a and two runs of b is $f/t=90/3432=15/572$.

The method of generalizing this result is obvious and leads to the following theorem:

If the various permutations of N letters a and N letters b have the same probability of occurring, and if R_a and R_b denote the numbers of runs of a and of b, respectively, then the probability of R_a runs of a and R_b runs of b is given by

$$(2) \qquad k\frac{[(N-1)!]^2[N!]^2}{(R_a-1)!(N-R_a)!(N-R_b)!(R_b-1)!(2N)!}$$

where $k=2$ if $R_a=R_b$, and $k=1$ if $R_a\neq R_b$.

If $R_a=R_b$, we may start a permutation with either a's or b's. If $R_a\neq R_b$ (these numbers may differ by at most one), then the permutation must start with the letter having the larger number of runs. This is the reason for the two values of k. Now let u equal the total number of runs of a's and b's, i.e., $u=R_a+R_b$. It is clear that the probability for any admissable value of u may be calculated by using (2). For example, if $u=5$, then we must have either two runs of a and three runs of b, or three of a and two of b, so that the probability that $u=5$ is the sum of the probabilities of these two possibilities. If u is odd, there are always two pairs of values for R_a and R_b; if u is even, then there is only one pair of such values.

Now we can return to the question concerning marks. If a penny is thrown many times, and the throws are divided into sets of, say, 50 throws each, then the number of heads (or the per cent of heads) will vary from set to set. This will be true even if we always use the same penny so that the probability of throwing a head does not change. Similarly, even though your probability of failing a student (or your standard for failing work) does not change, still your per cent of failures will vary from semester to semester. These variations in the per cent of heads, or in the per cent of failing students, are called fluctuations of sampling. You wish to know whether your per cent of failures has remained constant except for the expected fluctuations of sampling. In other words, has your probability of failing a student, or your standard for failing work, remained constant from semester to semester?

To answer this question, we first find the median of your per cents of failure. This median is a number such that half of the per cents are smaller and half are larger. To find it we arrange your per cents in order of magnitude. If there is an odd number of per cents, the median is the middle one; if there is an even number of per cents, it is the mean, or average, of the middle two. Now taking the observed per cents in chronological order, we assign an a to each per cent above the median and a b to each per cent below the median. If there is an odd number of per cents, the median is one of the per cents, and we omit it in assigning a's and b's. In the resulting arrangement, or permutation, the number of a's will equal the number of b's, and this number is N. We count the total number of runs of a and runs of b. This is the number u. Suppose now there is an upward trend in your per cent of failures. This would tend to cause a cluster-

N	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$u_{.05}$	2	3	3	4	5	6	6	7	8	9	10	11	11	12

N	18	19	20	21	22	23	24	25	26	27	28	29	30
$u_{.05}$	13	14	15	16	17	17	18	19	20	21	22	23	24

ing of the b's at the beginning of the permutation and a clustering of the a's toward the end of the permutation. If there is a downward trend in your per cent of failures, this would tend to cause a clustering of the a's at the beginning and b's at the end. But any clustering of the letters would reduce u, the total number of your runs. If your u is so small that it is unlikely to have occurred as a fluctuation of sampling, that is, by accident, then probably you have a real upward or downward trend in your per cents and not just an apparent one due to the fluctuations of sampling.

Whether the trend is upward or downward is, of course, observable from the data. The difficult question is whether it is real or only accidental like the variation in the per cents of heads in sets of throws of a penny. To answer this question, it is customary to choose a $u_{.05}$ such that the probability that u is less than or equal to $u_{.05}$ is at most 0.05. To obtain this $u_{.05}$ we could calculate using (2) the probabilities of u equal 2, 3, 4, $\cdots$. We could then sum these probabilities in order, starting with $u=2$ and stopping when the sum was equal to 0.05 or when the sum was such that the addition of another probability would make it more than 0.05. The value of u corresponding to the last probability used would be $u_{.05}$. This procedure would be very tedious, and it may be avoided by the use of published tables, for instance, in Swed and Eisenhart: *Tables for Testing Randomness in a Sequence of Alternatives*, Annals of Mathematical Statistics, Volume 14 (1943), pp. 66–87, or in Wilks: *Elementary Statistical Analysis*, and elsewhere. A brief table of this kind, which will serve our purpose, is shown above.

If your u is less than your $u_{.05}$, then either a rare event with a probability of less than 0.05 has happened, or your data have been affected by an external cause, presumably a change in your standards in marking. We choose the latter alternative. If your u is greater than your $u_{.05}$, then there is no indication of a change in your standards in marking as far as your data are concerned.

In an actual case, over a period of 24 semesters, an instructor had u equal to a total number of 11 runs. Since there were 12 a's and 12 b's, N was equal to 12. From our table, $u_{.05}$ equals 8. Since 11 is greater than 8, there was no indication of a change in the instructor's standards in marking. Of course, a fraction other than 0.05 could have been used in the preceding discussion. However, 0.05 is a fraction commonly used for the purpose. Obviously the method could be used also for marks other than failure and for the marks of an entire department or an entire school.

Clerk in a bookstore: "This book will do half of your work for you."
Student: "Fine. I'll take two."

• DEVICES FOR THE MATHEMATICS CLASSROOM

Edited by Emil J. Berger, Monroe High School, St. Paul, Minnesota

Illustrating simple transformations

by William Koenen, Central High School, St. Paul, Minnesota

In compliance with a policy adopted by the Editorial Board of THE MATHEMATICS TEACHER that departmental editors who have served for six years be requested to relinquish their positions, this is the last section being submitted for publication. The department editor wishes to thank all those who have assisted in the publication of this department during the past six years.

—*E. J. B.*

Described in this note is a device which may be used to illustrate several simple transformations that can be applied to mathematical functions.

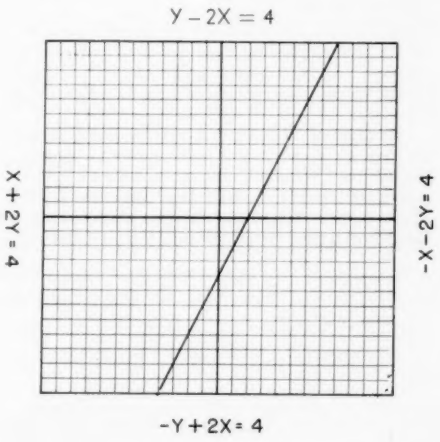
The device is easy to make. Scratch or paint a cartesian grid on one side only of a $\frac{1}{8}''$ sheet of transparent plexiglass. This is the entire device.

To use the device, plot and draw graphs with a grease pencil on the side that is not scratched or painted with the grid. If this direction is followed, all marks may be wiped off cleanly with a piece of cloth whenever a different graph is to be made. Colored scotch-type tape may be used if more permanent marks are desired. Plot graphs in the usual way, bearing in mind that markings will show off best if the background is sensibly chosen.

Since markings are visible from both sides of this device, one can use it to illustrate transformations that correspond to turning the graph to different positions or looking at it from the back side. To illustrate what we have in mind, suppose we graph the equation, $x+2y=4$, on the front of the plane. Now if we agree that the positive y-axis is always the axis that is "up" and that the positive x-axis is always "to the right," then we can, by turning the graph clockwise 90 degrees at a time, illustrate in succession the graphs of the following:

a) $x+2y=4$,
b) $-y+2x=4$,
c) $-x-2y=4$, and
d) $y-2x=4$.

Each equation in the list is obtained from the one which precedes it by the transformation, T_1 $(x, y)\rightarrow(-y', x')$.



The transformation which corresponds to turning the picture over (as one would turn the page in a book) may be expressed symbolically as follows: $T_2\ (x, y) \rightarrow (-x', y')$. Applying this transformation to equation d) above, we get,

e) $y+2x=4$.

Other graphs visible from the back are:

f) $x-2y=4$,

g) $-y-2x=4$, and

h) $-x+2y=4$.

As an additional suggestion it should be mentioned that one might develop the device described by making use of two transparent sheets—one for the grid, the other for the graph. If this plan is followed, it will be a simple matter to illustrate rotations and translations.

Thanks for work well done

The Editors of THE MATHEMATICS TEACHER wish to take this opportunity to thank several individuals for their past contributions to the work of getting out THE MATHEMATICS TEACHER. At the meeting of the Editorial Board held at Milwaukee last spring, the Board decided that, in general, it was good for the magazine to have departments "come and go." After all, there are a limited number of words that can be written about almost any subject, and the Board felt that an occasional new department, and even limitations on the amount of space devoted to departments, was good for the magazine. In accordance with these thoughts, "What's Going on in Your Schools?" and "Devices for the Mathematics Classroom" will not appear in the future issues of THE MATHEMATICS TEACHER.

Readers of THE MATHEMATICS TEACHER are grateful to J. A. Brown and Houston Karnes, Editors for "What's Going on in Your Schools?" and Emil Berger, Editor for "Devices for the Mathematics Classroom." The Editors take this opportunity to acknowledge their indebtedness and to thank Messrs. Berger, Brown, and Karnes for a job well done.

Other departmental editors found the pressure of duties and community obligations so heavy that they felt obligated to drop the work involved in getting out a departmental contribution eight times a year. In this class, we find Roderick McLennan, Co-Editor of "Reviews and Evaluations." We wish him good luck, and, above all, we want to thank him for doing a good piece of work for THE MATHEMATICS TEACHER.

Others who found the pressure of work too heavy and who should have been thanked long ago are Paul Clifford and Cecil B. Read. Paul Clifford's trip to Europe during the past year prevented his taking an active part in editing "Mathematical Miscellanea" and Cecil B. Read found it necessary to devote his time to other things. The Editors extend their sincere thanks to these two men for their assistance.

H. Van Engen, Editor
I. H. Brune, Assistant Editor

• HISTORICALLY SPEAKING,—

Edited by Phillip S. Jones, University of Michigan, Ann Arbor, Michigan

Irrationals or incommensurables IV—the transitional period

The earlier notes in this series told the story of incommensurables in Greek mathematics from Hippasus (500 B.C.) to Euclid (300 B.C.).[1] We saw how Eudoxus' theory of proportion for "magnitudes" suppressed the logical scandal caused by the discovery that there are geometric quantities or magnitudes whose ratio can not be represented by the ratio of two "numbers" where "number" at this time meant "integer." Eudoxus' theory made it possible to establish many synthetic geometry theorems without actually defining an irrational number. The need for a clear concept of *irrational numbers* as opposed to a theory of proportion for *incommensurable magnitudes* began to be felt with the development of analytic geometry and became essential as the theory of limits and continuity was developed with the nineteenth-century drive to give a rigorous foundation for the calculus.

This note is to summarize the status of incommensurables and irrationals in the transitional period between Euclid and Descartes (1637) or perhaps Cauchy (1825). This is a long period, and even though it is true that in it real developments with respect to irrationals were few and took place slowly, it is still a bit of an oversimplification to condense the entire period into such a short note as this. However, the chief growth in this period was in the development of computational devices for finding numerical approximations for roots of numbers and in operations with and manipulations of expressions involving radicals. Neither of these developments either represented or stimulated a recognition that here was an entirely new class of numbers which once defined and studied would be a mathematical system in its own right and not merely a device for treating ratios of "magnitudes." In fact the very words used implied a denial to these quantities of status as numbers.

Although we can see that the Eudoxian-Euclidean theory of proportion applies to all types of what we today call irrationals, Euclid used this theory only in situations where geometric magnitudes were involved, and in Book X treated only types which were readily obtainable by geometric construction and which in modern symbolism would be represented by $\sqrt{\sqrt{a} \pm \sqrt{b}}$. In fact the Greek words, which became, via Latin translations, our "rational" and "irrational," meant *expressible* and *inexpressible* in the original. Further, in Euclid, *irrational* was used only for quantities which were incommensurable "in square." Thus, although the diagonal of a square would be "incommensurable" with its side, it was not "irrational" in a Euclidean sense since $(\sqrt{2})^2$ is commensurable with $(1)^2$. Cassiodorius

[1] P. S. Jones, "Irrationals or Incommensurables," THE MATHEMATICS TEACHER, XLIX (February, 1956), pp. 123–27, (March, 1956), pp. 187–91 (April, 1956), pp. 282–85.

[2] Florian Cajori, *A History of Mathematics* (New York: Macmillan, 1918), p. 68.

(c. 585) is said to be the first to use "irrational" for the numbers to which it is now applied.[2]

Surd is a second word which is used confusingly even today and hence is probably best not used at all. This word came from the use of the Latin *surdus*, meaning silent or mute, to translate an Arabic word meaning "deaf" which the Arabs had in turn used in their translation of Euclid. The nonmathematical word *absurd* means "out of harmony" with reason; *illogical* has a similar origin in the Latin for "inaudible." Although *absurd* is not a mathematical word and *surd* did not come from it directly, it is interesting to note that it was first used in this sense in English by Robert Recorde in his *Whetstone of Witte* (1557) where he said, "8−12 is an absurde number. For it betokeneth less than nought by 4."[3]

In all usages $\sqrt{2}$ would be termed a *surd*, but there have been writers who would have excluded $\sqrt{6}$ because it can be written as $\sqrt{2}\cdot\sqrt{3}$, while many, such as Chrystal, would exclude $\sqrt{2+\sqrt{3}}$, reserving surd for the "incommensurable root of a commensurable number."[4]

It should not be thought that there was no progress at all from the Euclidean concept of incommensurable magnitudes toward a modern concept of irrational numbers in the years from 300 B.C. to 1800 A.D. Diophantos (*ca.* 250?) developed some algebraic symbolism and thought of numbers as distinct from geometric magnitudes in his work on theory of equations and number theory. He did not, however, accept either negatives or irrationals as numbers or as solutions to his problems. Hindu and Arabic writers recognized that quadratic equations have two roots which might be negative or surd. This led them to state rules for operating with surds.

For example, the Hindu Bháscara (*ca.* 1150) gave rules for the four fundamental operations with surds[5] which may provide some interesting enrichment material for a class. He said, "Term the sum of two irrationals the great surd; and twice their product the less one. The sum and the difference of them reckoned like integers are so [of the original surd roots]." He clarifies this with the example which we write with modern symbols as follows:

To find $\sqrt{2}+\sqrt{8}$, take the square root of the sum of the great surd, $2+8=10$, and the less surd, $2\sqrt{2\cdot 8}=8$. This finally gives $\sqrt{2}+\sqrt{8}=\sqrt{(2+8)+2\sqrt{2\cdot 8}}=\sqrt{10+8}=\sqrt{18}$. In symbols this says $\sqrt{a}+\sqrt{b}=\sqrt{a+b+2\sqrt{a\cdot b}}$ which today is easily verified by any student of second-year algebra and which will always suffice to write the sum of two square roots as a single radical, although the latter may also itself contain a second radical.

Bháscara's second rule for the sum and difference of two radicals was "the root of the quotient of the greater irrational number divided by the less, being increased by one and diminished by one; the sum and remainder being squared and multiplied by the smaller irrational quantity, are respectively the sum and difference of the two surd roots." Applying this to the first example, he wrote the equivalent of

$$\sqrt{2}+\sqrt{8}=\sqrt{\left[\sqrt{\frac{8}{2}}+1\right]^2\cdot 2}=\sqrt{18}.$$

We leave a literal formulation and verification of this to the reader.

Bháscara wrote similarly involved rules for the multiplication and division and involution of irrational expressions. However, although they were being operated with and were recognized as the solutions of equations, such expressions can hardly be said to have had the status of numbers.

[2] *The Oxford English Dictionary*, I (Oxford: Clarendon Press, 1955), p. 43.

[3] D. E. Smith, *History of Mathematics* II (Boston: Ginn & Co., 1925), p. 253.

[4] H. T. Colebrook, *Algebra, with Arithmetic and Mensuration from the Sanskrit of Brahmegupta and Bháscara*. (London: 1817), p. 145 ff.

[5] P. S. Jones, "The $\sqrt{2}$ in Babylonia and America," THE MATHEMATICS TEACHER, XLII (October, 1949), pp. 307–8.

Another development which we can now view as to some extent tending toward an idea of irrational number was the derivation of several processes for finding successive rational approximations to irrationals. Of this nature were the side and diagonal numbers and the related continued fraction expansion of the $\sqrt{2}$ which we cited in an earlier article. Of course, approximations to irrationals date back to Babylonian times, and mathematicians, such as Hero, in the later periods of Greek mathematics, developed repeated approximation processes which served practical needs. These processes shifted thinking from purely geometric considerations toward the arithmetical aspects of irrationals. However, since many persons regarded them as merely practical calculations which determined rational numbers which approximately satisfied the conditions represented by a symbol, these techniques, from another viewpoint, tended to turn attention away from the more basic and theoretical problem of defining a new kind of number.

Slight steps which from a modern viewpoint can be seen to be progress in the right direction are found in the work of Nicholas Chuquet in the thirteenth century who replaced the terms *square* and *cube* roots by *second* and *third* roots, and in the fourteenth century Nicholas Oresme simplified calculations with radical expressions by introducing fractional exponents. In the sixteenth century Michael Stifel made more substantial progress toward a general theory of irrational numbers by distributing numbers into different classes, by replacing lines by numbers in some of his discussions, and by formulating an analogy between "true numbers" and the symbols for irrationals. However, he still denied to the latter status as true numbers and remained attached to the ideas of Euclid.

In the same century Simon Stevin sought to show that roots and fractions have similar properties, and soon thereafter the development of the theory of equations and algebraic symbolism at the hands of Cardan, Viete, and Descartes accelerated the separation of number ideas from geometric interpretations as lines, areas, and volumes. This together with the calculation of logarithmic and trigonometric tables prepared the way for the treatment which the development of rigorous notions of limits and continuity required and which, as we will see in the final article of this series, grew out of the efforts of Cantor, Dedekind and others in the nineteenth century.

Bhabes Chandra Choudhuri, of Jalpaiguri, India, sent us copies of the two letters which follow. We print them because they reveal something of the patience, modesty, and desire to help others of a great physicist whose work also influenced the development of mathematics and philosophy and, indirectly, the daily lives of everyone on our planet.

BABUPARA, "MISRA-LODGE,"
JALPAIGURI (INDIA)
The 2nd. April, 1953

A minor sidelight on a great man:

Dr. Albert Einstein
Princeton University
New Jersey

Most respected sage of learning!

As I am in a fix in fully realizing the meaning of the NEWTONIAN Dynamics in so far as it relates to the interpretation of the movement of the planets, stars etc. in the space, and though am now in the teaching line and have often to put my young beginners through the daily grind of its mechanics, I must candidly own at your feet that my difficulties therein remain ever before as unrealized and still continue to confront my mind as mysteriously as the first Day in my

early village-School when I had to remember under fear of a birch-rod viz, that the earth which supports us, hangs and moves supportless in the void (space)!

Oh, Intellectual Nestor of the modern age!

May I therefore, most respectfully crave your attention to my following queries and hope to receive some light thereon from your august self just to set my uneasy mind at rest:

A. (i) How is it that our earth can hang and revolve supportless in the great "Void"—(in the word of the Bible) whereas, we must have some support to hang on?

(ii) If the earth is an inert mass, how can it get its energy from, for its so-called eternal auto-gyration in the space?

(iii) How can the earth perform the dual tricks of magical movements at the same instant in a non-stop way, viz, in its diurnal and annual phases?

(iv) How can it change its course constantly but retain *in situ* its unchanged constant obliquity?

(v) If the orbit or the ecliptic is an (imaginary) unreal thing, how can the real thing Equator make the angle of 23°: 28′ etc. with that?

I shall be also immensely grateful if my following query in Physics, receives your kindly illumination for an answer:

B. If it is a fact that iron is the real "mother" of electricity (because it is the core of the Dynamo) how is it that it should behave as the worst type of a "Conductor"?

With my great obeisance and profound respects to you and believe me, Professor,

I am most obediently, Yours,

BHABES CHANDRA CHOUDHURI

112 MERCER ST. PRINCETON, N.J.
May 25, 1953

DEAR MR. CHOUDHURI,

I have no time to answer your question exhaustively, therefore only a few remarks;

A. (1) If a body is alone in space, it could not possibly fall, for there is no privileged direction for such a process.

(2) To conserve motion by absence of friction, no delivery of energy is needed.

(3 and 4) Cannot be answered in a short way.

(5) The ecliptic is in the same sense real as the equator. One is determined by the axis of rotation of the earth, the other by the motion of the earth relatively to the sun.

B. Iron is not "the mother of electricity," but only the producer of the magnetic field, the latter being needed for the production of an electric current by magneto-induction.

With best wishes,

A. EINSTEIN

"Most important in the first place is the avoidance of rifts and controversy in the ranks of education itself. A damaging and in my opinion useless controversy has raged about the manner in which the high school system of the country fails to provide adequate preparation for college and university. It is quite true that every college without exception has to take care of some unfinished high school business. It is quite true that the problems of numbers, costs, and quality of instruction are made more difficult by this alleged defect or delinquency in secondary education. Somewhere we manage to lose at least a whole year and maybe more in the preparation of students who go on to college. But it is also true that in the system of secondary education in this country, preparation for college is only one of many educational functions. It is not even the most important. The pattern of secondary education is set for our generation. The meaning of this is clear. The responsibility of dealing with the deficiencies of the high school graduate remains with the colleges and universities. It cannot be lessened by quarreling with the high schools, nor met by the premature and intemperate severity with which insufficient preparation is often punished in the first two years of college. If it is true that we are approaching a serious shortage in skilled and highly qualified manpower in the nation, then we cannot afford to flush out potential talent before it has had time to prove itself."—*Taken from an article by Cornelis W. de Kiewiet,* How Different Types of Institutions Are Planning for the Future, *in* Action Under Way, *January 1956. Published by American Council on Education, Washington, D.C.*

• MATHEMATICS IN THE JUNIOR HIGH SCHOOL

Edited by Lucien B. Kinney, School of Education, Stanford University, and Dan T. Dawson, School of Education, Stanford University, Stanford, California

Sequoia Junior High School Math Team

by Paul H. Hofeditz, Sequoia Junior High School, Fresno, California

The Sequoia Junior High School Math Team is now in its third year. It was originally organized as a demonstration group consisting of hand-picked students of outstanding mathematical ability. They were to develop skills in rapid calculation and learn a number of dramatic short-cut "tricks" with which to impress and amaze other students and lay groups.

During the three years of its existence, the Math Team has undergone several changes, both in its purpose and in the type of student participating. Membership is now open to any student wishing to make a hobby of mathematics, regardless of previous success in arithmetic. At the present time we have 26 seventh-, eighth-, and ninth-grade students with a wide range of ability.

The activities of the group have expanded. We still stress skill in number manipulation and the learning of short-cut methods. Each student, however, must earn the right to use any particular short cut by demonstrating a thorough understanding of the mathematical principle involved.

The public demonstrations, of course, are the activities most enjoyed by the students. A demonstration group usually consists of five students; one to present the problems and four students who work them on portable blackboards. They are willing and anxious to explain the number theory back of the methods they used, and they are particularly proud to demonstrate speed and accuracy in the fundamental processes.

So far this year, in addition to demonstrations for the other arithmetic and algebra classes in our own school, Math Team members have appeared before an Air Corps Reserve Officers group and the P.T.A., performed during the intermission at the annual city-wide Mathematics Tournament, and appeared on a local TV program; they have also performed for many visitors at the school. There has been no lack of volunteers for these performances.

The group meets every day during the "activity period" (first period in the morning). That its membership is as large as it is, is amazing in view of the fact that it is not a "play period" and it is competing with some 15 other activities such as an outstanding choral group; science and photo clubs; modern dance; cadets; tumbling; art, craft, and shop hobbies; baton corps, etc.

The major problem has been maintaining interest five days a week throughout the school year. There is a limit to what can be done with drill and short-cut methods. By trial and elimination the following activities have become a standard part of the program:

1. Comparison of our number system with others such as the Roman, Egyptian, and Mayan, and the use of the counting board, abacus, etc. This leads to discussion and study of number systems with bases

other than ten, and makes the understanding of the theory of mechanical computors a little easier to grasp.

2. Each short-cut method is analyzed, expanded, and sometimes adapted to fit a variety of situations. While no really *new* short cuts have been worked out, the following illustrates what can be done:

> Beginning with the short method of multiplying numbers such as 67×63, the students independently recognized its value in squaring any two-digit number ending in five. The square root of numbers resulting from this multiplication can be found by reversing the method. A combination of this process with the short method for multiplying numbers such as 57×63 resulted in a short method for multiplying any pair of two-digit numbers together and, in turn, a quick method of multiplying *any* number by a two-digit multiplier, etc.

Not of earth-shaking importance, but the thinking involved required not only interest, but also a thorough understanding of positional value in multiplication.

3. A large chess-board, made of butcher's paper mounted on cork, is a permanent fixture in the classroom. Checker and chess men, cut from manila paper, may be moved about by means of pins so that the entire group may watch the play. Interested students learn the elements of the two games and are encouraged to work out their own strategies. Chess problems are taken from books, newspapers, and magazines and are presented to the group for solution. The emphasis is on seeing ahead as many moves as possible. This same faculty is applied to outlining, orally, the sequence of processes involved in the solution of rather complicated arithmetic problems.

4. A "problem of the week" is presented on Mondays. Usually these are brought in by students. Quite a few have been contributed by parents attempting to stump the youngsters. After these problems have been worked by arithmetic, the ninth-grade algebra students on the team enjoy demonstrating the ease with which some of them may be solved using algebraic methods. Naturally the seventh- and eighth-grade students are looking forward eagerly to studying algebra.

5. "Puzzle" problems, both ancient and modern, are very popular, especially if their solution requires mathematical reasoning or the logical organization of information. "Trick" problems are good for a laugh, but hold no lasting interest. Cryptography and magic squares appeal to some.

6. Science fiction and the problems of space travel are of great interest to this age group. Frequently an entire period is devoted to this subject. One result is that, like Congressmen discussing the national budget, large numbers no longer frighten them.

7. Interesting reports on the lives of great mathematicians and scientists are presented. Child prodigies seem to be the most fascinating subject in this field.

8. The most recently adopted activity is the attempt to solve a variety of arithmetic problems using only a ruler and a right-angle triangle. This was not too successful at first, but after some practice in scale-drawing, followed by simple problems in area, volume, and per cent, it seems to be catching on.

Since the basic purpose in organizing the group is to interest these young people in studying more mathematics, there is a strong temptation to give the abler students more than an inkling of higher mathematics. We feel that such a thing would be a mistake. Everything is done to stimulate their interest and curiosity concerning trigonometry, geometry and calculus; nothing is done to blunt this by giving them "simplified introductions" to these subjects.

The Math Team program is still being developed. We hope that it will continue to expand and improve each year. Judging by reports on former members now attending high school, we feel that we are on the right track.

• MEMORABILIA MATHEMATICA

Edited by William L. Schaaf, Department of Education, Brooklyn College, Brooklyn, New York

On the usefulness of mathematical learning (1700 A.D.)

Bernard Le Bovier de Fontenelle was secretary of the Royal Academy of Sciences at Paris during the latter part of the seventeenth century and the early part of the eighteenth century. In the year 1699 he wrote a *Preface to the Memoirs of the Royal Academy, treating of the Usefulness of Mathematical Learning*. An abridged English translation of this unusual essay appeared in *Miscellanea Curiosa*, Volume I (London, 1723). We reproduce here a few excerpts from this translation, not so much because of what is said concerning the "usefulness" of mathematical study, but rather for the remarkable clarity with which it portrays the dramatic spectacle of the world of science poised on the doorstep of a new century. The breathless expectancy of undreamed possibilities, on the very threshold of new and unknown horizons, has been captured in these few pages with rare fidelity.

BUT to what purpoſe ſhould People become fond of the Mathematicks and Natural Philoſophy? Of what Uſe are the Tranſactions of the Academy? Theſe are common Queſtions, which moſt do not barely propoſe as Queſtions; and it will not be improper to clear them.

People very readily call uſeleſs, what they do not underſtand. It is a ſort of Revenge; and as the Mathematicks and Natural Philoſophy are known but by few; they are generally look'd upon as uſeleſs. The reaſon of this is; becauſe they are crabbed and not eaſily learnt.

This is the Fate of Sciences, which are ſtudy'd and improv'd by few. Moſt People are not ſenſible of their Progreſs, and eſpecially when made in ſome mean Callings. But what doth it ſignifie, that we can now more eaſily direct the Courſe of Rivers, cut out Canals, and ſettle new Navigations; becauſe our Method of taking the Level and making Sluces is infinitely better than heretofore? Some Maſons and Seamen have thereby found their Buſineſs eaſier, but they themſelves were not ſenſible of the Skill of the Geometrician who directed them. They were mov'd, as the Body by a Soul, it doth not know how. Others are yet leſs ſenſible of the Genius that preſided over the Undertaking; and the World is the better for its ſucceeding well, but not altogether free from Ingratitude.

Altho' the Uſefulneſs of Mathematicks and Natural Philoſophy is obſcure, yet it is real. To conſider Mankind in their Natural State, nothing is more uſeful to them, than what may preſerve their Lives, and produce thoſe Arts, which are both great Helps and Ornaments to publick Societies.

One wou'd think at firſt, that if the Mathematicks were to be confin'd to what is uſeful in them, they ought only to be improv'd in thoſe things, which have an immediate and ſenſible affinity with Arts, and the reſt ought to be neglected as a vain Theory. But this wou'd be a very wrong Notion. As for Inſtance, the Art of Navigation hath a neceſſary Connexion with Aſtronomy, and Aſtronomy can never be too much improv'd for the Benefit of Navigation. Aſtronomy cannot be without Opticks by reaſon of Perſpective Glaſſes; and both, as all other Parts of Mathematicks, are grounded upon Geometry, and to go as far as you can, even upon Algebra.

Geometry, and eſpecially Algebra, are the Keys of all the Inquiries, that can be made concerning Magnitude. Theſe Sciences which are only converſant about abſtruſe Relations, and ſimple Ideas, may ſeem dry and barren, whilſt they keep within the Verge of the Intellectual World; but mixt Mathematicks, which ſtoop to Matter, and conſider the Motion of the Stars, the Augmentation of moving Forces, the different Paſſages of the Rays of Light through different Mediums; the different Effects of Sound by the Vibration of Things; to conclude, all thoſe Sciences, which diſcover the particular Relations of ſenſible Magnitudes, go on farther and more ſecurely, when the Art of diſcovering Relations in general is more perfect. The Univerſal Inſtrument cannot be too extenſive, too handy, or too eaſily apply'd: It is uſeful to all the Sciences, and they cannot be without it: And therefore among the Mathematicians of the Academy, who

are deſign'd to be uſeful to the Publick, the Geometricians and Algebriſts make a Claſs, as well as the Aſtronomers and Mechanicks.

However, it is certain, that Speculations purely of Geometry, or of Algebra, are not about uſeful Things: But it is certain too, that thoſe that are not, either lead or belong to thoſe that are. It is in it ſelf a very barren thing to know, that in a Parabola a Subtengant is double the correſponding Abſciſſæ; but yet it is a Degree of Knowledge neceſſary to the Art of throwing Bombs, ſo exactly as they can do now. There are not by far ſo many evident Uſes as Propoſitions or Truths in the Mathematicks: Yet it is enough if the Concourſe of ſeveral Truths is generally of ſome uſe.

Farther, a Geometrical Speculation, which was not at firſt applicable to any Uſe, becomes ſo afterwards. When the greateſt Geometricians in the Seventeenth Century ſet about to ſtudy a new Curve, which they call'd a Cycloide, they only engag'd themſelves in a meer Speculation out of Vanity, ſtriving to out-do one another by the Diſcovery of difficult Theorems. They did not even pretend that this was for the Publick Good; however, by diving into the Nature of the Cycloide, it was found, that it was deſtin'd to make Pendulums as perfect as may be, and carry the Meaſure of Time as far as it can go.

Let us always make a Collection of Mathematical and Phyſical Truths; happen what it will we can't hazard much by it. It is certain, that they ſhall be drawn from Springs, whence a great many uſeful ones have already been drawn. We have reaſon to preſume, that we ſhall draw from thence, ſome that ſhall ſhine as ſoon as they are diſcover'd, and convince us of their Uſefulneſs. Other Truths ſhall ſtay ſome time till a piercing Meditation, or ſome happy Accident diſcovers their Uſe. Some Truths being conſider'd by themſelves ſhall be barren, 'till they are conſider'd with reference to one another. Laſtly, let the worſe come to the worſe, ſome ſhall be eternally uſeleſs.

I mean uſeleſs with reference to ſenſible and groſs Uſes; for otherwiſe they ſhall not be ſo. An Object upon which alone you caſt your Eyes is the clearer and brighter, when the neighbouring Objects, which however you do not look upon, are alſo enlighten'd; becauſe it hath the Benefit of the Rays, which are reflected from them. Thus thoſe Diſcoveries, which are palpably uſeful, and deſerve our chiefeſt Attention, are in ſome meaſure enlighten'd by thoſe, which may be call'd uſeleſs. For all Truths make one another more lucid.

It is always uſeful to have right Notions, even of uſeleſs Subjects.

A Geometrical Genius is not ſo confin'd to Geometry, but that it may be capable of learning other Sciences. A Tract of Morality, Politicks, or Criticiſm, and even a Piece of Oratory, ſuppoſing the Author qualify'd otherwiſe for thoſe Performances, ſhall be the better for being compos'd by a Geometrician. That Order, Perſpicuity, *Preciſion* and Exactneſs, which ſome time ſince are found in good Books, may originally proceed from that Geometrical Genius, which is now more common than ever, and in ſome manner is communicated by one Relation to another, nay, even to thoſe that do not underſtand Geometry. Sometimes a Great Man draws all his Contemporaries after him; and he who hath the juſteſt Claim to the Glory of having ſettled a new Art of Arguing, was an excellent Geometrician.

Laſtly, whatever raiſes us to Great and Noble Reflection, tho' they be purely Speculative, afford a Spiritual and Philoſophical *Utility*. The Wants of the Mind are perhaps as many as thoſe of the Body. She deſires to extend her Knowledge: All that can be known, is neceſſary to her, and there can be no better Proof than this, that ſhe is deſign'd for Truth. Nothing perhaps can redound more to her Glory, than the Pleaſure that is felt ſometimes, in ſpight of ones ſelf, in the dry and crabbed Queſtions of Algebra.

But without running counter to the common Notions, and recurring to Advantages which may ſeem too far-fetch'd and refin'd, it may fairly be own'd, that the Mathematicks and Natural Philoſophy have ſome things which are only ſubſervient to Curioſity . . .

We submit that these excerpts, in addition to having quaintness and historical significance, also carry a moral or two which mathematics teachers might well heed: to wit, that one never knows to what "practical" use a new development in "pure" mathematics may one day be put, and further, that the study of mathematics has always been pursued "for its own sake," i.e., intellectual curiosity *per se*. Both aphorisms should definitely be brought home to our students.

Observations on the prestige of mathematics. . . .

On at least two occasions,[1] Professor I. I. Rabi, noted physicist at Columbia University, has pointed out the low esteem in which modern science is currently held. Deploring the fact that both the government and the public are insensitive to the

[1] See, for example, the *New York Times*, January 2, 1956, p. 27; also, *New York Times* Magazine, February 12, 1956, p. 14.

impact of science on culture, Rabi goes on to say:

> As the importance of science in the country increases, its dignity seems to be diminishing. Many scientists have had the frustrating experience of trying to explain what science is about to laymen, either in government, or in the universities, or to the ordinary professional or business man.
>
> Such is the spirit of the time that it is difficult, if not impossible, to communicate the feeling of dedication and reverence which all physicists have for our discipline. They do not seem to understand the emotional commitment we have to expand and deepen our understanding of nature. Nor does the quest seem to be particularly important except under the aspect of the conquest of nature.

According to Rabi, scientists and scientific knowledge in the last decade have gradually come to be regarded by the populace primarily as national assets, indispensable for the military strength of the country, invaluable for extending the resources, conveniences, and amenities of life, and generally raising the standard of living. Thus

> What disturbs and frightens the scientist is the increasing tendency to treat science and the scientist as a commodity. . . . The great drive now going on to increase the number of scientists and engineers takes on the appearance of stockpiling of tungsten or copper.

Virtually the same observations can in truth be made with respect to mathematics. Ask the man on the street whether he believes that mathematics is important. If he has thought about it at all, he will point at once to its value in the art and techniques of modern warfare. Such prestige as mathematics may have in the eyes of the lay public almost certainly devolves upon its strategic relation to the scientific applications involved in defense and military preparedness. Together with the implications of applied mathematics for the industrial and economic aspects of contemporary civilization in times of peace, these two major considerations comprise the sole "defense" for the case of mathematics.

How tragic! Not a hint of the human desire to cultivate a discipline for its own sake; of the satisfaction of sheer creation; of the impact of rigorous thinking upon successive cultures. Two passages come to mind at once; would that they were emblazoned on the bulletin board of every mathematics classroom in the land. Both passages appeared in the Catalogue of St. Johns College in Annapolis.[2]

> Three hundred years ago algebra and the arts of analytic mathematics were introduced into European thought by René Descartes. This is perhaps the greatest intellectual revolution in recorded history, paralleling the other great revolutions in religion, morals, politics, and industry. No liberal, and therefore no citizen of a democratic country, can afford to be ignorant of this change and its issues. It has redefined and transformed our whole natural and cultural world.
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> * * * * *
>
> Next to the mother tongue the language of numbers and figures is the most important symbolic possession of men. In fact it is a language within the mother tongue providing a most powerful practical and theoretical extension. In view of our present scientific and industrial conditions of life, the decay and elimination of mathematics in education is most disturbing. This default has become so common now that many persons believe that they natively lack mathematical ability. Nothing could be more crippling to the individual nor more discouraging for the future of democratic societies, if it were true.

Guided missiles. . . and mathematical education. . .

How timely and appropriate is some of our current mathematics teaching? Is it attuned to the world of today? Consider the field of guided missiles and the related problems of inertial navigation. How many mathematics teachers are familiar with the principles of inertial navigation? An uninhabited missile hurtling through space at an unprecedented speed—with no clues whatever from the earth or stars to guide its flight—it uses the take-off point as basic reference, "senses" all accelerations which act upon it, and moves

[2] *Bulletin*, 1945–46, St. Johns College, pp. 21, 26.

through a "universe" that consists only of space, time and acceleration. The "components" which make this possible consist of *accelerometers* to measure accelerations; *integrators* to convert this information into velocity and distance; *gyroscopes* to provide directional reference and hold the system stable; and additional *computers*, besides the integrators, to calculate course-to-steer and distance-to-go.

What kind of knowledge is required for work in this field? These men must be experienced in *systems evaluation, gyroscopics, digital computers, accelerometers, telemetry, guidance systems, stabilizing devices, servomechanisms, automatic controls, thermodynamics, optics,* and *environmental research.* To find out more about the kind of mathematical training and the mathematics involved in guided missile work, we wrote directly to one of the leading firms in this field; their reply is quoted below, by permission:[3]

> . . . a mathematician in the missile field faces a variety of very interesting and highly challenging mathematical problems. Due to the rapidly advancing missile technology most of the problems do not fall into categories presently treated in textbooks. Instead, the missile mathematician must rely upon keeping abreast of the technology by voluminous reading of the most recent literature on the subject. Of even greater importance, he must draw upon his own training, ingenuity, and resourcefulness as a means of successfully attacking and solving the very complex problems.
>
> A mathematician working in the guided missile field is primarily called upon to work with ordinary and partial differential equations. In addition, he should have a thorough knowledge of analytical geometry, calculus, and functions of real and complex variables, as well as a thorough understanding of differential equations. A good background in general physics, especially theoretical mechanics, is also extremely helpful.
>
> Some technical people in the missile field have very definite ideas on the early and advanced training requirements necessary for a mathematician to enter this industry. They advance the opinion that students at college level are now being taught primarily by the "memory" method by which they commit to memory facts and conclusions without really understanding them. Instead, the technical people say, the training should stress basic understanding or native ability and ingenuity of the student, thus making him think and reason rather than rely upon what he has been told as fact and committed to memory.
>
> The professional missile mathematicians also point out that the mathematical training at the high school level might be improved. They point to low salaries and resultant low social standing of the high school mathematics teacher as being important factors in the caliber of high school training. The solution seems to be an adequate pay scale for the mathematics and science teachers at least approaching that being paid by industry for similar educational background and experience.
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> In their concern for obtaining properly trained mathematicians to fill the requirements in the missile field, a few technical people in the industry advocate a stringent means of separating the bright high school student from those less scholastically inclined. In principle they would like to see, after a given number of years, of elementary schooling, students given competitive examinations to indicate those capable of continuing on for a higher education. Those who pass the examinations would enter secondary school, the others remaining a few more years in elementary school and then entering apprenticeship training to become skilled workers. Again at the secondary school level competitive examinations would be held and the same process followed. This method, the technical people believe, would allow students to pursue the type of training peculiar to their individual talents. Even more important, it would permit the brighter students to proceed much more rapidly in their education and preclude the possibility of being held back by less talented students.
>
> Generally, it is of prime importance that students be taught at the earliest possible level of education to use their ingenuity and resourcefulness and to maintain their curiosity and reasoning ability rather than rely upon theories and principles, which he is able to parrot and not really able to understand.

[3] Lockheed Aircraft Corporation, Missile Systems Division, Van Nuys, California.

A column of unofficial comment

National Council news

by Howard F. Fehr, President,
The National Council of Teachers of Mathematics

May I first express my gratitude to the members of the Council for the honor and confidence given to me in the election to the office of President. With the help of the other officers and members of the Board of Directors, I shall do my best to promote the interests of the Teachers of Mathematics and to carry on the fine program initiated by my predecessors in this office.

Recently, as was to be expected, there has been much excitement and controversy over the existing conditions in mathematics education, especially at the secondary school level. The curriculum has been criticized as outmoded and also defended as basic provided it is actually well taught. The teaching of mathematics has been attacked as being dull, unmotivated and poor; at the same time many teachers and even whole departments of mathematics have been publicly commended for results of their fine work. Teachers have been accused of not knowing their subject matter and yet many industries have employed high school and college mathematics teachers in positions calling for high command of the subject.

In all this debate, the National Council of Teachers of Mathematics has made no public statement, but it has quietly, through its several committees begun a calm, scientific, and scholarly study of the problems. It will continue this study with the hope that there may come a series of reports and conclusions that can lead to an improvement in mathematical education over present conditions. Our point of view is that we must not criticize, condemn, or blame others for the present state but seek a positive program of action that promises better educational conditions along with the possibility of successful administration.

Several facts are patent. First there is a tremendous lack of teachers *qualified to teach mathematics*. In New York City more than half of all the high school classes in mathematics are manned by teachers "out of license," that is, persons who have not been certified by passing the regular examinations as teachers of mathematics. This same condition is reported all over the U.S.A. To aggravate this situation, almost daily, the better mathematics teachers are leaving teaching to enter industrial positions paying higher salaries. How can we get more teachers of mathematics?

In the second place, in this emergency, qualifications, especially as to knowledge of subject matter, are being lowered. Many states and cities are issuing emergency teaching licenses to prospective persons not having sufficient subject matter background with the hope that the licensee will study more mathematics. In these cases the amount or kind of courses to be pursued for a permanent license is seldom specified. Foundations are offering summer courses of real value, but the teachers attending are for the most part those who need this work the least. How can we protect mathematics instruction by securing proper training and high certification requirements for all teachers of the subject?

In the third place, our curriculum, while

not bad in comparison with other countries, has not kept pace with the creation of new mathematics and with the changes occurring in the collegiate programs and in the applications of mathematics. Mathematical concepts are not static but change as new and improved structures of mathematical thought are evolved by present day mathematicians. Our present program, with few exceptions, is the same program introduced into the high schools from 1890 to 1900 when the secondary school population was quite different in scholastic characteristics and when much of contemporary mathematics was not yet in existence. How can we create a program in mathematics which is adapted to the wide range of mental capability that exists in the high school and which is also in harmony with contemporary thinking in mathematics?

The study of this last problem is now under way by the Curriculum Committee. The ultimate goal of this committee is the publication of a report that will have real impact on curriculum in mathematics. This committee will cooperate with the several other committees of other organizations who are also engaged in studying various aspects of the mathematics curriculum.

To study the first two problems mentioned above, so that the National Council can take genuine leadership in directing mathematics education in the elementary and secondary schools of our country, it is proposed to establish a Committee on Teacher Training, Certification, and Recruitment. All the members of the Council are invited to take part in the responsibilities and personnel activities.

In the interim, until this committee is appointed and gets to work, the Council is fortunate in having the committees on International Relations, on Research, on Talent Search and numerous other projects, keeping it well informed on important happenings in mathematics. The Council is indebted to the many busy people who give so graciously of their time, energy, and money to further the education of our youth.

Mathematics in our modern industrial economy

"Most of us, however, still lack full realization of how deeply mathematical techniques and modes of thought are penetrating into our everyday lives. The intrusion of mathematics everywhere differs in one important respect from the manifestation of mathematics in the past. Yesterday, mathematics appeared as a mere helper, a subservient handmaiden of science and engineering. Today mathematics is taking a role of leadership and appearing in applications devoid of prior organization in scientific form.

. . . . . .

Let us attempt to point out the reasons inherent in mathematics which might explain this widespread power and usefulness in all systematic inquiry. Mathematics offers a medium for expressing complicated and logical relationships. It provides a framework for organizing masses of data and information. Its concepts and symbols display a dynamic dependence of the performance of the whole upon the behavior of its parts."—*Taken from a publication of the Industrial Mathematics Society* Mathematics in an Industrial Economy.

Reviews and evaluations

Edited by Richard D. Crumley, Iowa State Teachers College, Cedar Falls, Iowa

BOOKS

Going Places with Mathematics, M. Peters. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1956. Cloth, vii+358 pp., $3.22.

Although not specifically so stated, it would appear that this book could best be used in grade nine with the non-mathematics students. The content is built around a family trip. The children are high-school freshmen. While it is primarily arithmetic there is work on simple equations, statistics, angles, areas, volumes and the tangent ratio. There are many good pictures and illustrations which contribute to the instruction.

The explanations are good but do not always help in understanding. For example precision of measurement is considered in all problems but no reasons why. Tables of squares, tangents and other tables are given but no emphasis is given to the basic structure of tables.

The content is adequate and the vocabulary is appropriate for the poorer reader. It seems the author has invented some words for this group such as "ruler fractions," page 19.

As is true of most books, there are a few minor errors such as on page 17 where the illustration and explanation do not agree, and on page 63 where approximate measure and rounded estimates are confused.

There is a great deal of story problem material about things that would be likely to interest such a group of students. The appendix contains basic arithmetic tests and practice material. If this material is wisely used by the teacher in relation to the text, it will be very helpful in raising the students' level of operation.—*Philip Peak, Indiana University*

Math. Can Be Fun, Teacher edition, Louis Grant Brandes, Portland, Maine, J. Weston Walch, 1956. Paper, iv+200 pp., $2.50.

This mimeographed collection of recreational mathematics items is probably unique in that all the material has been used successfully in classes. The collection is characterized by its variety. Number oddities, puzzles, number stories, tricks, magic squares, and many other types of material are included. A section devoted to "cross-number" puzzles is noteworthy. The book includes a useful bibliography of forty-six other publications offering similar material. Answers are provided wherever possible. An important feature is a section devoted to suggestions to teachers for getting the maximum instructional value from recreational mathematics in the classroom. A very comprehensive index completes the work.—*Harry E. Benz, Ohio University, Athens, Ohio*

Through the Mathescope, C. Stanley Ogilvy, New York, Oxford University Press, 1956. Cloth, vii+162 pp., $4.00.

There seems to be a constant need and a constant audience for books which present mathematics from a popular standpoint and require little background in mathematics. This collection of mathematical ideas should appeal to the clever high school student, the wide-awake teacher, the non-mathematical college student, and all curious adults. It treads the middle ground between a book of puzzles with tenuous connections to important mathematics, and the opposite, but still popularized treatment, which tries to develop mathematical ideas in a logical sequence. It does neither of these, but does its own job very well.

Unfortunately, the rest of the book does not live up to the promise of the first chapter where it says that it will "point out how some of the apparently disconnected topics are related." It is true that in the chapter on "Pi and the Pi-makers" we see π as the limit of a converging alternating series, as the limit of an infinite product, an infinite radical expression, a continued fraction, the result of integrating to find the area of a circle, a probability (in the Buffon needle problem), the area under a normal curve, the log (-1), and then we hear about rational approximations to π and something of circle squarers and angle trisectors. Surely this is a fascinating series of topics and well chosen, appealing to all the audiences noted above and practically useful to teachers who wish to have supplementary reading or projects grow from good books. However, the phrase "how these topics are related" might lead some people to expect a description of logical connections between remote topics, and not merely, as we do find, a simple juxtaposition of topics with eyebrows raised in amazement that the same arithmetical constant keeps appearing.

Other chapters have just as interesting an assortment of facts and ideas as those listed for the chapter on π. Some of the other chapters are concerned with numbers, probability, conic sections, calculus and calculus of variations, and topology.

A special word should be said for the lighthearted style in which the book is written. It is

very difficult to write mathematics with style without becoming quaint or whimsical. Without being too obtrusive, the style implies that mathematics really is fun for the author and will be for you, too.—*Henry W. Syer, Boston University*

Carl Friedrich Gauss: Titan of Science, G. Waldo Dunnington, New York, Exposition Press, 1955. Cloth, xi+479 pp., $6.00.

Biographies of mathematicians can be both helpful and entertaining. In general such biographies lean toward either a popular style which ends by being entertaining, or toward a scholarly style which is ultimately helpful to other scholars. This book aims at being scholarly. Outstanding biographies manage to do both.

Scholarship consists of diligence, accuracy and insight. The present biography of Gauss passes the first two tests but fails on the third. There is no doubt that a great deal of time and a life of devotion to the subject have accumulated as many facts as one could ever wish about Gauss; these are displayed with dates and details that convince the reader of accuracy. For example, in appendices we find such standard information as a bibliography of works by and about Gauss and a chronology of his life; and we also find more unusual material such as a list of the courses which he taught at the University of Gottingen, a list of the books which he withdrew from the library while a student, and a list of his descendants (with information about their spouses) to the present, eighth, generation. Moreover, the text itself seldom mentions an event without trying to tell the exact day on which it occurred. Birth and death years are included for almost all references to people to whom Gauss wrote regularly or with whom he came into professional contact. Such details may make the book a source of information for all future generations, but they certainly kill all chance of its having a lively, interesting, readable style. It almost seems as though the mountain of facts, once collected, was too precious for even a shovelful to be removed.

More seriously, a biography must communicate a sense of a person who lived. There was always the chance that the piling up of small details might have resulted in a pointillistic portrait, but it did not. Much of this failure is due to the rigid selection of only those facts which pertain directly to the subject of the picture. Should not a biography paint background as well, telling of the atmosphere of German universities, the type of social life of the period, some small hints of custom, costume, and manners so that we can see this mathematician as a man living in the world of his day? Obviously, the task set for himself by the author is to include only that information which can be found in letters and official records and to avoid all suspicion of speculation. I submit the opposite view that the duty of a biographer who knows as much about his subject as this author, undoubtedly more than anyone else in the world, is to select the pertinent facts, and to conjecture about and to interpret the mind of the man he portrays. Why did Gauss do what he did? How did his mathematical mind work? What were others doing at the same time, in fields other than mathematics and science, which affected the life and feelings and work of Gauss? What effect did Gauss's work have on the other sciences? What effect on the world as a whole? Even though only guesses, the guesses of one who is worth hearing.

For example, we hear that Gauss was a gentle man but that there were a half-dozen people who disliked him very much. Even if we cannot be sure, a conjecture on his personality, on the methods or achievements which created the antagonism of such men as Abel, Jacobi, Legendre and Halstead would be revealing. Admittedly, this is difficult for one who brings awe to his task of writing a biography.

It seems as though this might be compared to a mosaic as well as a painting. We are shown a tremendous number of separate glass bits, all lovingly collected and now displayed in a huge picture of a brilliant rainbow. We admire the skill with which the colors have been matched and carefully arranged to produce such a large and scientific picture. Another artist could have used the same pieces to create the portrait of a man and a mathematician.—*Henry W. Syer, Boston University*

Applied General Mathematics, Edwin B. Piper, Randolph S. Gardner, and Preston E. Curry, New Rochelle, New York, South-Western Publishing Co., 1954. Cloth, viii+566 pp., $2.60.

As the name implies, the emphasis of this text is on applications, chiefly to business and industry. A great many business forms and procedures are presented.

Arithmetic processes and new problems are introduced by way of the business forms in which they are used. This should prove a useful motivating device. Very little is done to give meaning to the various processes, however. In most instances a rule is stated, an exampled solved according to the rule, then further exercises are listed for the student. The meaningful development which general mathematics students particularly need will have to be supplied by the teacher.

The text contains an abundance of graded problems and drill materials. In addition to cumulative reviews, drills for accuracy and speed are included. Achievement tests and a workbook are available. A limited amount of space is allotted to diagrams and pictorial material.

The unit of algebra and the one on geometry are too brief for the meaningful development of the basic ideas and concepts presented.

In my opinion this text could be useful for a terminal course in consumer mathematics either as a basic text or as a reference text for the business and financial forms and procedures.—*Lee Dulger, Thornton Township High School, Harvey, Illinois*

Using Mathematics, Kenneth B. Henderson and Robert E. Pingry, New York, McGraw-Hill Book Co., 1955. Cloth, xvi+555 pp., $3.36.

The authors attempt to include as many concepts as possible into this textbook. The subject matter varies from simple arithmetic processes through elementary algebra, geometry and trigonometry. Ample practice material is given and the language used seems to be simple enough for most high school students.

Assignments are the same for all students with no provision for faster or slower students. This reviewer feels that the individual teacher can better provide for these differences than can a textbook, and that the above is therefore a point in favor of the book.

Informal geometry and the use of drawing instruments is the first topic of this text. This somewhat different approach to the subject might help overcome the feeling on the part of many students that the high-school general mathematics program is a repeat of eighth-grade arithmetic.

In the chapter on decimal measuring there are a number of measuring devices illustrated. Among these can be found the ammeter, voltmeter, pressure gage, tachometer, altimeter, dial gage, vernier, micrometer, and various special gages (go-no go gages, snap gages, etc.) This gives a practical approach to the use of decimals.

There is a good unit of 40 pages on graphs. This chapter concerns both the reading and construction of graphs. There is a question, however, about the reason for including mean and median in this particular chapter. This topic is suddenly introduced in the middle of the chapter and then just as suddenly is dropped.—*Carl T. Uth, Arlington High School, Arlington Heights, Illinois*

Functional Mathematics, 7th Grade, Beulah Echols, William A. Gager, Franklin W. Kokomoor, Richard Madden, and Carl N. Shuster, New York, Charles Scribner's Sons, 1955. Cloth, viii+390 pp., $2.24.

This book appears to be a worthy contribution to the series published by Scribner's. The authors have commendably presented elementary mathematical concepts in a variety of situations that should arouse student interest and promote learning. Problems are constructed about situations that are appealing to children. Building, cooking, farming, nature lore, and sports are a few of the areas included within this text. The drawings are pertinent, appealing, and abundantly displayed. Titles and subtitles describe the forthcoming concepts in an interesting manner that should assist the development of student interest and awareness.

Although *Functional Mathematics, Grade 7* has many fine features it has some questionable ones as well. There is within the current criticism of mathematics a criticism of over-simplification of subject matter. Such simplification tends to prevent students from making greater advances. It appears to the reviewer that this book, as with many good texts, can be criticized for such a weakness in its presentation of certain topics. Admittedly it is best to write to the level of the seventh grade youngster, but it seems that too frequently books written to the verbal level of the reader fail to increase his ability to comprehend more advanced terminology. Consider this statement regarding geometry appearing on page 241: "A study of lines, angles, and the size and shape of things is called geometry." Will an incomplete statement such as the preceding adequately apply to geometry? Can it and related statements give the student an awareness of the properties of proper expression, and can it describe geometry as adequately as a more concise statement? Could a better word than "things" be used to classify geometric figures?

The tests and practice exercises that constitute Chapter XII appear to be well thought out, although they may be lacking quantitatively. No guide is provided within the chapter that will allow the student to know what he should study to correct weaknesses indicated by his failure to work certain problems properly.

The positive features of this text far outweigh those that appear questionable to this reviewer.—*Roderick C. McLennan, Arlington High School, Arlington Heights, Illinois*

Trigonometry With Tables, A. M. Welchons and W. R. Krickenberger, Boston, Ginn and Co., 1954. Cloth, vii+419 pp., $3.20.

This high school text might be briefly described as a thorough, carefully worked out, conservative treatment of the subject. It contains everything that almost any traditional teacher could ask for. However, in reflecting the spirit of modern mathematics and making use of modern learning theories, the book seems to this reviewer to be deficient. It would seem that study of the book might lead a student to skillful manipulation and problem solving of a certain sort, without producing any very deep mathematical understanding.

The approach is through the functions of an acute triangle. Since it is assumed that many students have already had an introduction to the trigonometry of the right triangle, this treatment is very brief. The reader is told the definitions of the various ratios, is given a set of exercises applying these, and is *then* presented with a proof that the sine of an angle does not depend upon the size of the triangle; this non-developmental approach is rather typical of the psychology which the authors seem to use. Approximate numbers and logarithms (Chapter II) and the slide rule (Chapter III) receive maximum application throughout the rest of the text. The exposition of approximate numbers seems confusing and overly dogmatic; the topic of significant figures is poorly treated, as illustrated by the exercise asking for the number of significant figures in 186,000 miles, an impossible task without some context for this measure-

ment. The un-mathematical practice of not clearly labelling definitions is especially aggravating here in connection with everyday words such as "error" and "accuracy"; it is very difficult to tell which sentences are merely explanatory and which are definitions giving these words their highly technical meanings. Scientific notation is briefly explained but is applied neither to the matter of significant figures nor to the possible improvement of the usual rules for finding logarithms.

The usual topics of the general angle, graphs, identities, the oblique triangle, and so forth, are treated in a thorough, standard fashion. The section on plane trigonometry concludes with a chapter on complex numbers and the hyperbolic functions, and this is followed by thirty pages on spherical trigonometry.

Exercises seem adequate in number and scope. There are "A" exercises for all students and "B" exercise for extra work, of a more challenging sort. The motivational value of the applied problems is doubtful; problems about observation balloons or about the area of a boathouse roof are not likely to be very exciting to students of the mid-twentieth century and it seems a shame not to show some of the applications to modern science and technology. Directions on "how to remember" this or that and the numerous procedures displayed in boxes will be viewed as good or bad according to the individual teacher's philosophy. The only errors noticed were of a foundational nature and perhaps of little consequence; for example, the statement in the chapter on complex numbers (page 277) that "We can use the letter i just as we use any other letter in algebra." The photographs and drawings seem helpful and worthwhile; physically, the book is well printed and presents a very attractive appearance.—*Robert S. Fouch, The Florida State University, Tallahassee, Florida*

Editor's Note: In addition to single reviews of two or more books by the same reviewer or reviewers, occasional books will be reviewed by more than one reviewer. See review of this book written by Edward Spacapan in this issue.

Trigonometry With Tables, A. M. Welchons and W. R. Krickenberger, Boston, Massachusetts, Ginn and Co., 1954. Cloth, vii+419 pp., $3.20.

The text seems to follow a trend in appealing to the aesthetic reaction of an individual. Color contrast in the text is eye catching in its use of red and black. Red is used in denoting subtitles, examples, in graphs, triangles, and in some of the figures. The color could have been used more effectively if it were used only to stress the mportant features of the book. It could have been used in pointing out the definitions and rules instead of denoting the numerous subtitles.

The book spends some time on the derivation of the trigonometric formulas. Following each derivation, the book supplies a sufficient number of examples which the students could follow easily.

At the close of each chapter there is a review and at the end of Chapter 12 there is a general review. The book supplies enough exercises to bring the student to an understanding of the trigonometric relationships, but is short on problems involving equations of higher degree and identities or quantities in which only one member is given, allowing the student to simplify the problem as much as possible.

The book offers chapters on the slide rule, graphing of functions, inverse functions, complex numbers and hyperbolic functions, and three chapters on spherical trigonometry.

The text is concise and well written. The better students can follow the text without much help from the teacher. The authors put forth a great effort to supply ample examples in order to give the students a thorough working knowledge of the problems. There is a particularly good chapter on graphing, in which there is an application of the slope of a curve, amplitude, and periods of functions.

The work on the slide rule and logarithms could be omitted very easily if students have had previous experiences with them.

Five-place tables of logarithms and logarithms of trigonometric functions, and four-place tables of trigonometric functions are included at the end of the text. A list of all the geometric theorems necessary to an understanding of the trigonometry is included in the introduction.—*Edward Spacapan, Arlington Heights High School, Arlington Heights, Illinois*

Editor's Note: In addition to single reviews of two or more books by the same reviewer or reviewers, occasional books will be reviewed by more than one reviewer. See review of this book written by Robert S. Fouch in this issue.

Elementary Algebra, William G. Shute, William E. Kline, William W. Shick, LeRoy M. Willson, New York, American Book Company, 1956. Cloth, viii+488 pp., $3.20.

Algebra and Its Use, Book I, Ethel L. Grove, Anne M. Mulliken, Ewart L. Grove, New York, American Book Company, 1956. Cloth, vi+454 pp., $3.20.

Algebra, First Course, John R. Mayor and Marie S. Wilcox, Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1956. Cloth, viii+392 pp., $3.08.

For the purposes of this review, the above three books will be referred to by their titles; namely, *Elementary Algebra*, *Algebra and Its Use*, and *Algebra, First Course*.

The above texts are criticized from the standpoint of modern algebra. It is well to face the cold fact that algebra, as taught in our high schools, often has little in common with modern mathematics. Algebra is conceived of today by those who know it best as a beautiful structure: algebra is today a study of logical structures, it is not a hodge-podge of rules, definitions and skills. Because of this the reviewers feel that

the algebra books for the schools of today should be authored from the point of view of the structure of the subject which the student will ultimately understand in its full bloom if he continues to study mathematics. These reviewers have a strong faith, that the beautiful precise formulation of concepts which characterizes modern abstract algebra can guide us to a simpler and more vigorous presentation of the topics of high school algebra. Furthermore, it is the reviewers' belief that textbooks published in 1956 should give evidence of the impact of modern algebra on the algebra of the high school, even the algebra for the ninth grade. These books, in particular *Elementary Algebra* and *Algebra and Its Use*, do not reflect the changes that are needed in the approach to the fundamental concepts of algebra in the high school.

It is the reviewers' contention that algebra will be easier to learn if it does reflect the demands of the modern algebra taught in advanced college courses. Let us give an example. *Algebra and Its Use* describes variables by stating; "The quantities whose values change from one situation to another, such as p and s or t and r are called variables." There is real reason to believe that this group of words says little or nothing to the learner, and it certainly is not at all acceptable to anyone who has studied a moderate amount of mathematics. Compare this definition with the one initiated by developing the idea of a set (collection) of numbers. Suppose that this set of numbers is under consideration:

Set S: {0, 1, 2, 3, 4, 5, 6, 7, 8}

Now it becomes desirable to talk about the members of this set. Adopt the convention of using a symbol, such as x, to represent any member of the set. Then construct sentences involving this mark x. We may say: For every number x in S it is true that $x \leq 8$; there exists at least one number x in S such that x is even and $x > 3$; for every x, $x(x+1) = x^2 + x$; no x in S has more than 2 distinct prime factors. Out of this experience comes a rather clear understanding of variables; that is, a *variable* is a *symbol* used to represent an arbitrary member of a set.

The definition of a variable given above is a "modern" definition; that is, it is a definition which needs only minor "polishing" to make it acceptable to most algebraists. The reviewers would defend this definition, not only because it is mathematically acceptable, but also because of psychological advantages. The traditional definition is a jumble of words. The suggested definition supplies the insight necessary to any student who wants to be moderately successful in understanding what algebra is really all about. The pupil can almost see himself selecting numbers from the set and plugging them into a sentence to see if they "fit".

In the judgment of the reviewers the authors of *Elementary Algebra* and *Algebra and Its Use* have made a number of mathematical errors. An error is bad enough, but the tragedy of the situation is that mathematical errors often lead to even worse psychological errors. Correct mathematics is often much easier to learn than incorrect mathematics. In the specific comments to follow the reviewers have pointed out some of the things that they feel should not occur in algebra texts at the ninth grade level. The careful reader will be able to find others.

In general the reviewers are agreed that in the hands of a skillful teacher each of these three texts would be useable. All are attractive books. Each text presents much more material than can be covered by the average class in the first year of algebra. The inexperienced teacher will probably need some guidance in selecting topics.

Elementary Algebra: The material presented in this text is quite adequate for a one year course. We note a single exception: Only two units (Chapters 7 and 12) are devoted to word problems. While these units are excellent it would seem that more of such material should be available to the teacher.

The illustrative examples are well chosen. The combination of red with black type is quite pleasing and is very effective in calling attention to concepts which should be emphasized. The "flash backs" are good, adding interesting historical sidelights to the study.

Numerous small aids to the student are scattered through the text, but it is doubtful that the "boxed" rules have value. A rule without meaning is useless and that meaning can only be established by discovery and application.

The aids to the solution of "word problems" are commendable. In the development of equations for the solution of verbal problems use of the initial letter of the magnitude sought (w for weight, t for time, etc.) gives the letter x a much needed rest. The standard terms "transposition" and "cross multiplication" are employed. This is regrettable, for most students are led by this terminology into meaningless mechanical processes.

Most of the complaints to be levied against *Elementary Algebra* have to do with mathematical errors and the employment of terminology which, if learned by the student, will need to be unlearned in later mathematics courses.

Unfortunately the phrase "literal number" is employed. It would be splendid if beginning algebra texts started the important task of classifying the numbers of the complex number system as *natural numbers, integers, rational numbers* and so forth. Instead of referring to "literal" numbers it would be better to indicate by a wealth of illustrations how letters are used in algebra. When a letter is employed as a variable the student should always be informed as to the domain of the variable. At no time does *Elementary Algebra* make clear what specific set of numbers is the reference set for a particular variable.

No satisfactory definition of *factoring* is presented. It should be made clear that we al-

ways factor a polynomial relative to a chosen coefficient domain. In *Elementary Algebra* the coefficient domain appears to be the ring of integers although this is never stated. Prime integers and prime polynomials are not correctly defined.

In Chapter 4 the negative integers are introduced in standard fashion without definitions, hence, necessarily none of their properties can be proved. It is regrettable that the authors do not point out that if a is any rational number, then $-a$, called the negative of a, is the number such that $a+(-a)=0$.

On page 113 directions read: "Solve for the unknown letter." Why not solve for the unknown *number?*

On page 152 a ratio is called the "division" of one "quantity" by another. On page 344 a ratio is said to be the "comparison of two numbers by division." Nowhere is the modern point of view expressed, namely, that by the ratio of number a to number b we mean the number we get by dividing a by b. The vague terminology still used today in the discussion of ratio is over 2500 years old. We inherit it directly from the Greek mathematician Eudoxus who had no real number system.

The concept *square root of a number* is handled confusingly. The authors state, "5 is the square root of 25. Also, -5 is the square root of 25." Moreover they assert, "x^3 is the square root of x^6 and $-x^3$ is the square root of x^6, so every positive monomial has two square roots." What is a positive monomial? It is never pointed out that $\sqrt{a^2}=a$ if $a\geq 0$ and $\sqrt{a^2}=-a$ if $a<0$.

The language, "if $(x-2)(x-3)=0$, then $x=2$ *and* $x=3$," is used. Most teachers probably prefer the $x=2$ *or* $x=3$ terminology.

On page 291 occurs the remark, "We never speak of the *solution* of a linear equation in two variables." These reviewers often do so.

In Chapter 17 there is a discussion of the irrational number concept. We learn on page 364 that "the exact value of $\sqrt{3}$ can not be found." At last it becomes clear why our college freshmen make some fantastic remarks. They are quoting their high school texts accurately! If the authors had proved that search as we might we could never find $\sqrt{3}$ among the set of rationals, then we should accord them warm praise. Many students who use these texts will have the privilege of seeing the number $\sqrt{3}$ *constructed* in a college mathematics class.

On page 371 we find the statement, $\sqrt{5b}\cdot\sqrt{5b}=\sqrt{25b^2}=5b$. This gives deep insight into the often observed phenomenon that college freshmen can calculate $\sqrt{2}\cdot\sqrt{2}$ because they know $\sqrt{4}=2$, but they are completely helpless when faced by $\sqrt{783}\cdot\sqrt{783}$, since they can not extract the square root of 613,089.

Algebra and Its Use: This text provides a wealth of material. Selection of material to be used would be difficult for inexperienced teachers. Deletion or curtailment of some units would be necessary. In general the organization is good. The chapter on ratio and proportion might well come earlier in the text, while the difficult work on radicals could find a place nearer the end of the book.

Written problem lists are very good. Most of the applications are pertinent and within the range of student experience. Examples are generally adequate but may in some cases need to be supplemented.

The use of blue type with the black to emphasize explanatory work seems unfortunate. It does not stand out nearly so well as the red used in *Elementary Algebra*. All in all, however, this is a usable, teachable text designed for the use of able students.

Much of the terminology employed in *Algebra and Its Use* is objectionable. Definitions which are at best unsatisfactory descriptions occur frequently. For example, on page 81 we read: "A positive number is a number, often written with a plus sign before it, which represents a value greater than zero or an increase in value."

On page 133 an unfortunate term is used. The numbers 20 and 1 are called "roots" of the equation $x+y=21$, and the authors remark that such an equation has infinitely many roots. Strictly speaking this does not contradict *Elementary Algebra* since there it is held that $x+y=21$ has no solution. These reviewers tell their students that the equation $x+y=21$ has *infinitely many solutions* each of which is an ordered pair of numbers. The root terminology is reserved for equations in one unknown.

On page 282 reference is made to "negative expressions." We ask: which of the expressions $4-x$ and $x-4$ is the negative one? It is never made clear that when *negative a* is used as an abbreviation for *the negative of a* we are using the word *negative* in a sense altogether different from the way we employ it in the phrase *negative number*.

The assignment of meaning to such symbols as x^0, x^{-1}, etc. is done in logically incorrect fashion on page 334. It is unlikely that students will be greatly damaged by this presentation, but it seems unfortunate that they are given no opportunity to learn what is really happening. The treatment of square roots is careless. On page 359 occurs the statement "this gives $\sqrt{x^2}=\pm\sqrt{49}$. . . ." On page 342 we are told casually that "An even root of a negative number is always *imaginary*, . . . " This is the only reference to the complex number system. What purpose does this statement serve?

We close with a few quotations from the Glossary:

> *Integral expression:* a whole number expression.
>
> *Irrational number:* a number whose exact value cannot be expressed.
>
> *Directed number:* a number with a sign to indicate the direction in which it differs from zero.
>
> *Ratio:* the comparison of two quantities expressed as a fraction.

It is quite clear that if students learned textbook definitions well in high school mathematics, teaching college mathematics would be even more difficult than it actually is.

Algebra, First Course: This text differs markedly in several respects from the other two. It is heartening to observe that generalizations and rules are not set before the student long before he is psychologically prepared for them. The authors indicate that they are attempting to make it possible for teacher and pupils to employ the "discovery" approach. Accordingly we find in this text that very few formal definitions occur. Rather the authors content themselves with pointing at certain mathematical entities and indicating certain of their properties. Such descriptive techniques are certainly to be preferred over the listing of definitions which are meaningless jumbles of words.

These reviewers doubt that the exploratory activities presented in the text will actually lead in all cases to the discoveries and generalizations desired. And of course the authors are not completely consistent with their avowed aim to provide opportunities for pupil creativeness. The presentation of linear equations seems still formal and rule-bound. Little opportunity for discovery is there.

Sometimes the text treatment seems almost too casual. For example, negative numbers are introduced with the remark $5+(-3)=2$. The explanation on page 13 that $-3+4-2$ means the same thing as $(-3)+(+4)+(-2)$ is quite cursory. The experienced teacher will be able to supply needed emphasis, but the inexperienced teacher may fail to stress important principles which are treated lightly and informally in the text.

A discussion on page 18 concerning the use of letters in algebra is excellent. The description of polynomials on page 63, although not precise, is an improvement over the usual fare.

Many lists of review exercises are closely related to arithmetic. These are quite good. Treatments of per cents and decimals are especially to be commended. Elementary topics from geometry, trigonometry and analytic geometry are interwoven skillfully into the text. A supplementary chapter includes an interesting variety of topics and provides some additional work for superior students. The reference material suggested at the end of each chapter is well chosen and might prove helpful to a beginning teacher.

Relatively few criticisms may be levied against the terminology employed. On page 77, in discussing the multiplication of 32 by 21 reference is made to multiplication by "two" rather than *twenty*. No clear definition of a *factor* of an integer or polynomial is given. Again, no consideration of the coefficient field enters into the discussion of the factorization of polynomials. We find on page 221 the comment, "the factors of x^2-y^2, are $(x+y)$ and $(x-y)$." Is not every number other than zero also a factor? Moreover, what about the polynomials x^2-y^2, $3x+3y$, $\frac{1}{3}x-\frac{1}{3}y$, $2y-2x$, etc.?

As an introduction to an extension of the exponential notation to include the use of zero and negative integers as exponents, computations with integral powers of 10 are considered. This section provides a fine background for understanding the definitions we choose to make as we extend the exponent concept.

The treatment of quadratic equations is well done. However, the remarks on page 380 relative to the derivation of four equations from the equation $x^2=4$ by taking both positive and negative square roots seem meaningless. Why not emphasize logical reasoning rather than mechanical manipulation? Is it so hard to reason thus:

If $x^2=4$, then either $x=2$ or $x=-2$?

In summary we might say that the first two texts mentioned in this review are attractive books with excellent problem lists, but they provide little opportunity for pupil discovery. Emphasis is still upon meaningless mechanical manipulation. Moreover, these books contain much objectionable terminology. Sometimes this terminology leads to mathematical errors. In many other cases it furnishes meaningless statements. It would be well to ask a group of competent algebraists to review the terminology used in present day texts and suggest modifications. This would provide valuable aid for text book writers.

The third text seems to be an honest attempt to avoid the most flagrant errors which have crept into high school algebra texts. At least lip service is given to the ideal of providing students with opportunities to reason logically and participate actively in the creation of mathematics. If, in the judgment of the authors, a precise formulation of the definition of a certain mathematical concept would serve no useful purpose (in view of the mathematical immaturity of beginning algebra students) then no definitions are made. Simple descriptions take their place.

Once more we stress our belief that experienced and skillful teachers will find any of these three texts a satisfactory source for problems and will be able to prevent the existing errors from doing any great damage to the mental equipment of their students.—*Charles Brumfiel, Ball State Teachers College, Muncie, Indiana* and *H. E. Fenimore, Central High School, Muncie, Indiana*

EQUIPMENT

Slated Globe, Cradle Mount (Model 00221), W. M. Welch Scientific Company, 1515 Sedgwick Street, Chicago 10, Illinois. Metal-slated, 12″ globe in metal cradle with equatorial ring, $27.50.

The globe is finished in black except for two small "polar" caps and can be removed from the cradle easily. The cradle is made also of metal with felt covered supports for the globe. The equatorial ring is attached permanently to the cradle. The ring is marked with a time scale,

a mileage scale, and a degree scale. The first two scales apply when the globe is used to represent the earth. The equatorial ring may be used also for drawing great circle arcs on any part of the sphere. The diameter of the globe and cradle is 16″ and the height is 13½″.

The globe and the cradle are both well constructed and function well together. Chalk marks show up clearly on the surface of the globe and may be erased easily with a soft cloth. The felt supports will partially erase chalk marks when the globe is rotated extensively but this is unavoidable with this type of mount. Some slated globes revolve about a fixed axis only and so do not blur chalk marks on the surface. This mount, however, allows the globe to be rotated about any axis, a feature that is frequently desired. This globe would be very useful for illustrating problems concerning spheres, the earth, navigation, and so forth.—*Richard D. Crumley*

MODEL

Inverse-Squares Illustration (Model No. 3535), W. M. Welch Scientific Company, 1515 Sedgwick Street, Chicago 10, Illinois. Plastic and metal model showing law of inverse squares, $10.00.

This model consists of a metal frame supporting three square sheets of imprinted plastic spaced approximately 6¾″ apart. Four elastic threads extend from the top crosspiece (each thread in a straight line) through two of the plastic sheets to the metal and plastic base. The overall dimensions are 7″×7″×21″ high. Four rubber feet provide support for the model. The top plastic sheet has one square imprinted on it 2″×2″, the threads passing through holes at each vertex. The second sheet has four 2″ squares imprinted on it, the threads passing through the four external vertices. The bottom sheet has nine 2″ squares imprinted, the threads terminating at the four external vertices.

The model is well constructed and can be used effectively in illustrating similar pyramids or similar triangles. Probably few teachers of mathematics will have enough use for this model to justify its purchase unless it is used also by science teachers in the study of photometric measures, and gravitational, electric, and magnetic fields.—*Richard D. Crumley*

Have you read?

Johnson, Paul B., "Stacking Colored Cubes," *The American Mathematical Monthly*, June-July 1956, pp. 392–395.

You all have seen and maybe played with a set of six cubes on which each face of each cube is painted in one of six different colors, and this has been done in the thirty possible different ways. As you know, the problem is to stack them so certain colors do or do not coincide.

This is an interesting and informative article on the generalization of this problem. It will appeal to your good mathematics students. It gives proof for three theorems. Your students will be interested to note that there are 67,260 different sets of eight number key cubes that will stack properly, that if a set of eight will stack they will do so in two, four, eight, or sixteen ways. The students will also be interested in the logic of the proofs and how the problem is attacked. I think this is a very good article for the gifted.

Keller, M. Wiles. "What Are You Teaching?" *School Science and Mathematics*, March, 1956, pp. 210–220.

We must never forget that today's high school mathematics students are tomorrow's mathematicians. The approach to mathematics of tomorrow is vastly different than was the approach of yesterday. Mr. Keller's article will be of help to those of us who are aware of tomorrow's needs and want help in how we may proceed. For example, have you tried to develop the ideas involved in the set of real numbers through the use of axioms? Have you considered the limitations of subtraction and division and how they were overcome? Did your class ever discuss how operations with fractions are controlled by basic axioms and that this control leads to the rule "invert the divisor and multiply? Do you stop to consider that if x is a number, then x^2+3x is a number? When they solve an equation do your students recognize that they are really proving a theorem and also its converse?

But read the article and I am certain you will find a new zest for teaching your ninth grade algebra class.

Mee, John F. "The Creative Thinking Process." *Indiana Business Review*, February, 1956, pp. 4–9.

As teachers of mathematics one of our objectives is to increase the level of thinking among our students. Even though this article is directed toward the businessman it will be wonderful reading for your students. I am sure you can help them find illustrations in mathematics to show the levels of thinking which Mr. Mee calls habit level, problem solving level and creative level. You will also be interested in the necessary steps involved in creative thinking. This short article is well worth everyone's reading.—Philip Peak, *Indiana University, Bloomington, Indiana*

• TIPS FOR BEGINNERS

Edited by Francis G. Lankford, Jr., Longwood College, Farmville, Virginia

Teaching the concept of cubic measure through the use of manipulative aids

by Jen Jenkins, Bethany College, Lindsborg, Kansas

Cubic measure, encompassing as it does the ideas of both linear and square measure, is a complex concept which requires much care on the part of the teacher for its development. Effective manipulation of cubes and much discussion of their properties will be of great value.

Inch cubes, made of construction paper or wood, should be handled and studied by the class. Counting the sides or faces, followed by numbering them consecutively from one to six, gives the class an acquaintanceship with the boundaries of the cube. Spreading of fingers across each face in order, one to six, and running of fingers along the edges, one to twelve, should show the class the part played by linear and square measure in the formation of the cube. The edges should be measured with a ruler marked in inches; the faces should be measured with the square inch. Such statements as the following should be written and studied: "The cubic inch has six faces and twelve edges. Each face is an inch square. Each edge is an inch long."

To show that the inch cube takes up space, the class should cup their hands and hold cubes within them. Estimating the number of inch-cubes contained in certain cups, boxes, and cans, followed by placing inch-cubes within these receptacles will aid the pupils in realizing that cubes fill up space. At this time the following statement should be added to the last previously made: "The inch-cube fills up space."

Readiness for the meaningful development of the formula for the volume of a box may be built when the pupil forms several layers of inch-cubes within the box. Wooden cubes are particularly good for this exercise. For instance, two layers of 24 inch-cubes each may be formed, each layer being composed of six rows of four cubes each. Other arrangements may be formed, such as three layers of 16 each, with two rows of eight cubes each making a layer.

To show the pupils how large a cubic foot is, a large cardboard box should be cut down and remade into a box that is a cube with a one-foot edge. As in the case of the inch-cube, the counting and measuring of the faces and edges should be done. The discoveries made should be written in sentence form for study. Small spaces, such as closets, spaces under the teacher's desk or under a stairway, may be estimated in terms of cubic feet and then measured as accurately as possible with the several cubic foot boxes which the class has made. If there are enough such boxes, volumes of different shapes could be built.

To develop the arithmetical fact that 1728 cubic inches make up one cubic foot, remove the top of one of the cubic foot boxes and cover the bottom with 12 rows of 12 inch-cubes each. Using 11 more such cubes, make a column 12 cubes high in one of the corners, explaining that each of the cubes in the column represents a layer of cubic inches. Then the pupils will see that 12 rows of 12 inch-cubes each form 144 inch-cubes in one layer and that 12 layers of 144 inch-cubes each form 12 times 144 inch-cubes or 1728 inch-cubes.

The size of the cubic yard may prohibit its inclusion in a classroom as a manipulative aid. Some teachers, however, have found room in a corner for a yard-cube constructed of heavy wrapping paper and several yardsticks. The pupils thus have a visual impression of the size of the cubic yard. The yard-cube can be made more understandable by engaging in activities such as those used with the inch and the foot-cubes. Again, the conclusions reached should be stated in words and kept for future reference.

To develop the idea that 27 cubic feet = 1 cubic yard, a cubic foot box should be placed on the bottom of the cubic yard in one corner. To lead the pupils to the desired conclusions, the following questions might be asked:

1. What do you see in the corner of the cubic yard?
2. How many of these foot-cubes would fit in a row?
3. How many rows would there be on the bottom?
4. How many foot-cubes would there be in a layer?
5. How many layers would it take to fill the yard-cube?
6. How many foot-cubes would fill the yard-cube?

A cumulative or accompanying activity to use as a summarizing device would be an exhibit on a table or elsewhere in the classroom, showing the inch-cube and the foot-cube, the latter having 144 cubic inches on the bottom. The following information—actually discoveries which have been made by the pupils themselves—should appear on placards near the proper cubes:

PLACARD I

1 layer of inch-cubes in a foot-cube = 144 inch-cubes
12 layers of inch-cubes in a foot-cube = 12 times 144 inch-cubes
12 layers of inch-cubes in a foot-cube = 1728 inch-cubes or,
1728 cubic inches = 1 cubic foot

PLACARD II

1 layer of foot-cubes in a yard-cube = 9 foot-cubes
3 layers of foot-cubes in a yard-cube = 3 times 9 foot-cubes
3 layers of foot-cubes in a yard-cube = 27 foot-cubes or
27 cubic feet = 1 cubic yard

PLACARD III

1728 cubic inches = 1 cubic foot
27 cubic feet = 1 cubic yard

Prove as much as you can

Robert R. Halley, Avenal High School, Avenal, California

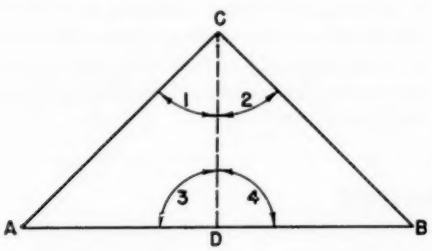
Here is a procedure which I have found useful in my teaching of geometry.* Instead of telling the students what to prove, I give them a drawing and an hypothesis, then ask them to prove as much as they are able. It is an elastic assignment which takes care of individual differences. The poor students don't prove much, while the bright ones fill pages. In this respect the students almost grade themselves on the assignment. I like the technique especially because it is a time-saver. A theorem and its chain of corollaries can be covered in one day. Otherwise two or three days are frequently used, or else the corollaries postulated.

Following are three examples, taken from work early in the year in a plane geometry course.

Example 1

Given: Any isosceles triangle ABC, with $AC = BC$, and CD the bisector of the vertex angle.

Prove: As much as you can.



Statements	Reasons
1. $AC = BC$	1. Given
2. CD bisects angle C	2. Given
3. Angle 1 = angle 2	3. CD bisects angle C
4. $CD = CD$	4. Identity
5. $\therefore \triangle ACD \cong \triangle BCD$	5. S.A.S. = S.A.S.

Theorem: The bisector of the vertex angle of an isosceles triangle divides the isosceles triangle into two congruent triangles.

6. $\angle 3 = \angle 4$
   6. Corresponding parts of congruent triangles are equal (C.P.C.T.E.)
7. $\angle 3 + \angle 4 = 180°$
   7. Two adjacent angles whose exterior sides lie in a straight line are supplementary.
8. $\angle 3 + \angle 3 = 180°$
   8. Equals may be substituted for their equals.
9. $\angle 3 = 90°$
   9. If equals are divided by equals, the quotients are equal.

Theorem: If two supplementary adjacent angles are equal, they are right angles.
Theorem: The bisector of the vertex angle of an isosceles triangle divides the isosceles triangle into two congruent right triangles.

10. Angle A = angle B
    10. C.P.C.T.E.

Theorem: The base angles of an isosceles triangle are equal.

11. $AD = DB$
    11. C.P.C.T.E.

Theorem: The bisector of the vertex angle of an isosceles triangle is the perpendicular bisector of the base.

12. CD is an altitude of $\triangle ABC$
    12. Definition of altitude

Theorem: The bisector of the vertex angle of an isosceles triangle is one of the altitudes of the triangle.

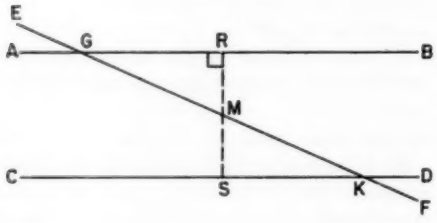
* I am indebted to Mr. John Schacht of Columbus, Ohio for first suggesting this procedure to me. It has worked so well in my classes, that I believe it is worth passing on.

13. CD is a median of $\triangle ABC$
13. Definition of Median.

Theorem: The bisector of the vertex angle of an isosceles triangle is one of the medians of the triangle.

Example 2

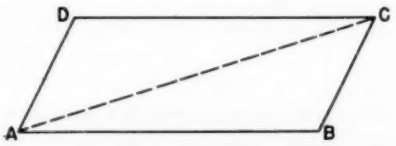
Problem: What can you prove to be true of the angles formed by the intersection of a transversal with two parallel lines? Suggestion: Through M, the midpoint of GK, construct RS perpendicular to AB. If you can prove triangle GRM congruent to triangle MSK, you are well on your way.



Without boring the reader with the details, the above exercise results in the proofs of all the theorems about the angles which are equal (or supplementary) when two parallels are cut by a transversal.

Example 3

Given: Any parallelogram $ABCD$
Prove: All you can about parallelograms
Suggestion: Draw a diagonal AC and prove as much as you can. Then draw the other diagonal and see if you can prove more.



Again, sparing the reader and the typesetter, I simply point out that this one assignment yields all the theorems about parallelograms usually considered by a plane geometry class. These, of course, are theorems that start with a parallelogram given.

I give one of these assignments toward the end of the class hour. The next day the students hand in their papers and the assignment is reviewed at the blackboard. I act as recorder and each member of the class is expected to contribute at least one step to the proof. By the end of the hour the many theorems that have been developed by individual students have been considered by the whole class. The students seem to enjoy this type of assignment as a relief from the more conventional exercises.

I believe there is much to recommend this type of assignment. It provides for individual differences; it is a time saver; it encourages original thinking in mathematics. Moreover, students enjoy this type of assignment as a change from routine, as well as the satisfaction of developing their own theorems.

Method—computation—answer

by Humphrey C. Jackson, Parcells Junior High School, Grosse Pointe, Michigan

For the past ten or more years I have used, at Grosse Pointe, Michigan, a method which I have found effective in improving the arithmetical solution of verbal problems by junior high school pupils. A regular-sized sheet of notebook paper is ruled into three columns by making a narrow column about one inch wide at the extreme right of the paper, and dividing the balance of the paper into two equal columns.

At the head of these columns the words *Method*, *Computation*, and *Answer* are written. The teacher then proceeds to

teach the pupils how to write out the solution of certain problems and to place the computation in the column provided for it. The method, using the symbols for addition, subtraction, multiplication and division, is carefully written out. When the solution is completed, the answer is written in the answer column.

This technique has been found to be particularly helpful in solving word problems. The following examples illustrate the method.

Example 1. *Problem:* A real estate agent sold a house for $12,500 at a commission of 5%. How much was his commission?

Solution:

Method	Computation	Answer
$C = 5\%$ of $12,500$	$12,500$	
$= .05 \times 12,500$	$\times .05$	
	625.00	$C = \$625$

This system was discussed at departmental meetings and several teachers tried using it. Eventually the junior high school mathematics department adopted this system for standardizing the presentation of the solution of verbal problems.

Those teaching algebra used the following variation in the solution of equations:

Solution	Check	Answer
$8x - 14 = 6 - 2x$	$8x - 14 = 6 - 2x$	
$8x + 2x = 6 + 14$	$8(2) - 14 = 6 - 2(2)$	
$10x = 20$	$16 - 14 = 6 - 4$	
$x = 2$	$2 = 2$	$x = 2$

It may seem that this system of organizing and presenting the work requires excessive writing, but we have found that after the pupil has used it for a while, he likes it and feels more sure of his work than before. We find that junior high school pupils like to have definite ways of doing things. They are able to follow directions if they are told exactly how to do things. This way of organizing their work does not solve the problem for the pupil, but simply helps them to be orderly and systematic about putting down the work.

We do not claim that this system is a cure-all for pupils with mathematical difficulties, but we do think it helps many pupils overcome careless habits such as doing the computation on scratch paper, throwing it in the waste basket, and then being unable to substantiate their answers by showing the necessary computation. It should be pointed out that pupils are urged to do all possible computation mentally, and to show on their papers only that which they cannot do mentally.

Freedom should also be given any pupil to discard this form as soon as he is able to do so. Independence of any particular pattern in solving problems is the ultimate goal of the pupil in studying mathematics. Such a method as I have described here should be regarded as a helpful learning crutch, but crutches should be discarded as soon as they are no longer needed. Bright pupils, especially, will become bored and may soon lose interest in mathematics when a pattern for solving problems is prescribed and continuously enforced

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Each pupil should have a textbook as a source of exercise material and data. Other textbooks and reference books should be provided. —*This statement is based on the study by William Lee Carter,* A New Basis of Organization for the Junior High School Mathematics Program. *Ph.D., 1952, Ohio State University, Columbus, Major Faculty Adviser; Dr. Harold F. Fawcett.*

What's new?

BOOKS

Secondary

Algebra, Book One, (rev. ed.), A. M. Welchons, W. R. Krickenberger, Helen R. Pearson. Boston: Ginn & Co., 1956. Cloth, xi+580 pp., $3.28.

Algebra, First Course, John R. Mayor and Marie S. Wilcox. Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1956. Cloth, viii+392 pp., $3.08.

Algebra In Easy Steps, (3rd ed.), Edwin I. Stein. Princeton: D. Van Nostrand Co., Inc., 1956. Cloth, vi+298 pp., $2.32 (paper, $1.56).

An Essay on the Foundations of Geometry, (Dover ed.), Bertrand A. W. Russell. New York: Dover Publications, Inc., 1956. Paper, 201 pp., $1.50.

Functional Mathematics, Book 4, William A. Gager, Luther J. Bowman, Carl N. Shuster, Franklin W. Kokomoor. New York: Charles Scribner's Sons, 1956. Cloth, xiv+578 pp., $3.40.

Plane Geometry, Rolland R. Smith and James F. Ulrich. Yonkers, N. Y.: World Book Co., 1956. Cloth, viii+536 pp., $3.84.

Solid Geometry, (rev. ed.), Daniel T. Sigley and William T. Stratton. New York: The Dryden Press, 1956. Cloth, x+197 pp., $2.60.

The Thirteen Books of Euclid's Elements, Volumes I, II, III, (2nd ed.), translated from the text of Heiberg with Introduction and Commentary by Thomas L. Heath. New York: Dover Publications, Inc., 1956. Paper, $1.95 each, set of 3—$5.85.

College

A Modern Introduction to Mathematics, John E. Freund. Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1956. Cloth, xvi+543 pp., $6.00.

Analytic Geometry, H. Glenn Ayre and Rothwell Stephens. Princeton: D. Van Nostrand Co., Inc., 1956. Cloth, viii+224 pp., $3.85.

Analytic Geometry, (2nd ed.), R. S. Underwood and Fred W. Sparks. Boston: Houghton Mifflin Co., 1956. Cloth, x+282 pp., $3.25.

A Short Table of Integrals, (4th ed.), B. O. Peirce, revised by Ronald M. Foster. Boston: Ginn & Co., 1956. Cloth, vii+189 pp., $2.25.

Calculus, Jack R. Britton. New York: Rinehart & Co., Inc., 1956. Cloth, xiv+584 pp., $6.50.

Calculus, William K. Morrill. Princeton: D. Van Nostrand Co., Inc., 1956. Cloth, x+537 pp., $6.00.

College Algebra for Freshmen, (2nd ed.), Gordon Fuller. Princeton: D. Van Nostrand Co., Inc., 1956. Cloth, vii+343 pp., $3.85.

College Algebra, (rev. ed.), Edward A. Cameron and Edward T. Browne. New York: Henry Holt & Co., 1956. Cloth, x+390 pp., $4.25.

Combinatorial Topology, Volume I, translated from the first Russian edition by Horace Komm, P. S. Aleksandrov. Rochester, New York: Graylock Press, 1956. Paper, xvi+225 pp., $4.95.

Differential Calculus, W. L. Ferrar. New York: Oxford University Press, 1956. Cloth, 296 pp., $4.40.

Differential Equations, (3rd ed.), Harry W. Reddick and Donald E. Kibbey. New York: John Wiley & Sons, Inc., 1956. Cloth, ix+304 pp., $4.50.

Elements of Mathematics, J. Houston Banks. New York: Allyn & Bacon, Inc., 1956. Cloth, x+422 pp., $5.75.

Engineering Applications of Vector Analysis, Leonard L. Barrett. Palo Alto, California: The National Press, 1956. Cloth, xii+114 pp., $3.50.

Famous Problems of Elementary Geometry, (2nd ed.), Felix Klein. New York: Dover Publications, Inc., 1956. Paper, xi+92 pp., $1.00.

Fundamental Mathematics, Thomas L. Wade and Howard E. Taylor. New York: McGraw-Hill Book Co., Inc., 1956. Cloth, xiv+380 pp., $4.75.

Fundamental Mathematics, Teacher's Manual, Thomas L. Wade and Howard E. Taylor. New York: McGraw-Hill Book Co., Inc., 1956. Paper, 53 pp., $1.00.

Intermediate Analysis, John M. H. Olmsted; New York: Appleton-Century-Crofts, Inc., 1956. Cloth, xii+305 pp., $6.00.

Introduction to the Theory of Groups of Finite Order, (Dover ed.), Robert D. Carmichael. New York: Dover Publications, Inc., 1956. Paper, xiv+447 pp., $2.00.

Irrationalzahlen, (2nd ed.), Oskar Perron. New York: Chelsea Publishing Co., 1951. Paper, viii+199 pp., $1.50.

Mathematical Analysis, Walter T. Hamilton and John Raymond Hamilton. New York: Harper & Bros., 1956. Cloth, xiii+379 pp., $4.50.

Mathematics of Finance, Robert Cissell and Helen Cissell. Boston: Houghton Mifflin Co., 1956. Cloth, ix+286 pp., $4.50.

Matrizenrechnung, Wolfgang Gröbner. Munchen, Germany: R. Oldenbourg, 1956. Pappeband, 249 pp., DM 23.

Partial Differential Equations, G. F. D. Duff. Toronto, Canada: University of Toronto Press, 1956. Cloth, x+248 pp., $6.50.

Plane Trigonometry, (2nd ed.), E. Richard Heineman. New York: McGraw-Hill Book Co., Inc., 1956. Cloth, xii+167 pp., $3.75.

Plane Trigonometry, with tables, (2nd ed.), E. Richard Heineman. New York: McGraw-Hill Book Co., Inc., 1956. Cloth, xii+242 pp., $3.75.

Practical Trigonometry, Ralph S. Underwood, Horace E. Woodward. Boston: Houghton Mifflin Co., 1956. Cloth, ix+251 pp., $3.25.

Principles of Mechanics, (Dover ed.), Heinrich Hertz. New York: Dover Publications, Inc., 1956. Cloth, xliv+44 pp., $3.50 (paper $1.75).

The Three R's Plus, edited by Robert H. Beck. Minneapolis: University of Minnesota Press, 1956. Cloth, x+392 pp., $5.00.

Trigonometrical Series, (2nd ed.), Antoni Zygmund. New York: Chelsea Publishing Co., 1952. Paper, vi+329 pp., $1.50.

MISCELLANEOUS

Aids and Shortcuts in Arithmetic, Ernst F. Bass. Portland, Ore.: Ernst F. Bass, 1956. Paper, 38 pp., $1.00.

Asymptotic Expansions, (Dover ed.), A. Erdélyi. New York: Dover Publications, Inc., 1956. Paper, vi+108 pp., $1.35.

Carl Friedrich Gauss: Titan of Science, G. Waldo Dunnington. New York: Exposition Press, 1955. Cloth, xi+479 pp., $6.00.

Electric Circuit Theory and Operational Calculus, (2nd ed.), John R. Carson. New York: Chelsea Publishing Co., 1953. Paper, ix+197 pp., $1.88.

Engineering Problems, Charles Angevine Hutchinson, Leon Watson Rutland, Jr., Walter Wayne Varner. New York: Harper & Bros., 1956. Cloth, vii+189 pp., $4.00.

Experiment and Theory in Physics, (Dover ed.), Max Born. New York: Dover Publications, Inc., 1956. Paper, 44 pp., $.60.

Free and Inexpensive Learning Materials, (7th ed.). Nashville, Tenn.: Division of Surveys and Field Service, George Peabody College for Teachers, 1956. Paper, viii+244 pp., $1.00.

Geometrical Optics, L. C. Martin. New York: Philosophical Library, 1956. Cloth, vii+215 pp., $7.50.

Going Places With Mathematics, M. Peters. Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1956. Cloth, vii+358 pp., $3.22.

How to Calculate Quickly, (Dover ed.), Henry Sticker, New York: Dover Publications, Inc., 1956. Paper, x+256 pp., $1.00.

Hydrodynamics, (Dover ed.), Hugh L. Dryden, Francis D. Murnaghan, H. Bateman. New York: Dover Publications, Inc., 1956. Paper, 634 pp., $2.50.

International Symposium on Precision Electrical Measurements. New York: Philosophical Library, 1956. Cloth, xxii+346 pp., $12.00.

Investigations on the Theory of the Brownian Movement, (Dover ed.), Albert Einstein. New York: Dover Publications, Inc., 1956. Paper, 121 pp., $1.25.

Rapid Calculations, A. H. Russell. New York: Emerson Books, Inc., 1956. Cloth, 287 pp., $2.95.

Risk and Gambling, John Cohen and Mark Hansel. New York: Philosophical Library, Inc., 1956. Cloth, x+153 pp., $3.50.

BOOKLETS

Advanced Placement Program, College Entrance Examination Board, 425 West 117th Street, New York 27, New York. Paper-bound booklet describing the new College Board program of advanced placement examinations, 136 pp., free to schools and colleges upon request.

Educational Requirements for Employment in Selected Professional Fields (Catalog No. VA 1.19: 7-8.1), Superintendent of Documents, U. S. Government Printing Office, Washington 25, D. C. 14-page booklet dealing with the requirements for actuaries, 15¢.

Lettering Techniques, The Visual Instruction Bureau, Division of Extension, The University of Texas, Austin 12, Texas. 33-page booklet prepared by Martha F. Meeks; $1.00 per single copy, discount for quantity orders.

Money Management: Your Equipment Dollar (Revised, 1955), Consumer Education Department, Household Finance Corporation, 919 North Michigan Avenue, Chicago 11, Illinois. 36-page illustrated booklet, 10¢.

Money Management: Your Shopping Dollar (Revised, 1956), Consumer Education Department, Household Finance Corporation, 919 North Michigan Avenue, Chicago 11, Illinois. 32-page illustrated booklet, 10¢.

Optical Illusions, J. Weston Walch, Publisher, Box 1075, Portland 1, Maine. 48-page assembly of stiff cardboard charts prepared by Louis Grant Brandes, $1.00 (cash with order).

Problems in Mathematical Education, Educational Testing Service, 20 Nassau Street, Princeton, New Jersey. Booklet written by Henry S. Dyer, Robert Kalin, and Frederic M. Lord, 50 pp., $1.00.

Teaching High School Mathematics, American Educational Research Association, 1201 Sixteenth Street, N.W., Washington 6, D.C. Booklet written by Howard F. Fehr for the *What Research Says to the Teachers* series, 33 pp., 25¢ for a single copy, discounts for quantity orders.

Understanding Numbers: Their History and Use, Ulrich's Bookstore, 547 East University Avenue, Ann Arbor, Michigan. Illustrated booklet written by Phillip S. Jones dealing with the same topics as the film set of the same name, 50 pp., $1.00.

CHART

Algebra on the Map, D. C. Heath and Company, Boston 16, Massachusetts. Chart, 17″×22″, identifying 20 persons or events in the history of the development of algebra, free while supply lasts.

EQUIPMENT

Basic Algebra Data-Guide, Student Marketing Institute, 375 Fifth Avenue, New York 16, New York. Permanent plastic sheet, 8½″ ×11″, printed on both sides with rules of procedure and formulas in basic algebra, holes at edge accommodate a two- or three-ring loose-leaf notebook, written by Myron F. Rosskopf, 69¢ each, prepaid, 42¢ each plus postage in lots of twelve or more.

Cube and Pyramid (*No. 598*), W. M. Welch Scientific Company, 1515 Sedgwick Street, Chicago 10, Illinois. Heavy-gauge transparent plastic models of cube and pyramid, each may be filled with water or sand to show ratio of volume of cube to volume of pyramid, $5.50 per set.

Slide Rule Data-Guide, Student Marketing Institute, 375 Fifth Avenue, New York 16, New York. Permanent plastic sheet, 8½″×11″, with holes which accommodate a two- or three-ring loose-leaf notebook, gives summary of rules and procedures for using a slide rule, written by Mario G. Salvadori and Jerome H. Weiner, 69¢ each, prepaid, 42¢ each plus postage in lots of twleve or more.

Space Spider, Walker Products, 1530 Campus Drive, Berkeley 8, California. Kit of three Masonite sheets with 225 holes in each, elastic thread, blunt needle, clips, and instruction booklet, for making three-dimensional geometric designs, $2.95.

Our changing world

Introduction

It is often been stated that the modern high school pupil does not possess ordinary skill in the common arithmetical processes. He has met unfavorable criticism in this respect not only from the college professor, who reports him unable to perform correctly the easy numerical work of the laboratory, but also from the man of business who frequently claims that the high school boy who finds his way into commercial life, cannot even add, subtract, multiply and divide.

As an experiment to test the truth of the latter criticism, the following questions, without previous warning, were simultaneously put to twelve hundred students in three of the large Philadelphia high schools, that is, to one hundred students in each of the four years of the course in each of the three schools.

The Question Paper

The question sheet was 9×14 inches, and the questions were properly spaced.

Philadelphia, Nov. 17, 1910

Read These Directions Carefully

1. Solve each problem in space provided, directly underneath.

2. Place all work on this sheet and indicate result clearly.

3. Write name and grade at top of reverse side of sheet.

1. Add the following:

34256	2. A man who has saved $142.96 still
78242	has $72.01 less than his brother has;
52167	how much have they together?
785862	3. A has 35¾ yards of cloth; B has 5½
1946	yards more than A; and C has 12¼
71234	yards less than B. How many yards
8761	have they all?
326504	
361	4. How much do I gain if I buy 300
651302	oranges at the rate of $1.25 per hundred, and sell at 25 cents a dozen?

5. From a farm of 800 acres I sell 10 per cent to Mr. Johnson, and 20 per cent of the remainder to Mr. Fleming. How much do I still own?

6. Multiply .052 by 108.04. Divide the product by 200. (Result to three decimal places.)

7. Find the interest of $800 for 9 years 5 months at 6 per cent.

It is evident that the above questions are in no sense a test of high school work, nor even of the advanced, or more difficult arithmetic of the elementary grades, but it is believed that they cover fairly well the more common computations of business. If the accusation of the business man is correct that our school product cannot perform the fundamental operations with proper facility, this test should surely show it. It was not claimed that it would show much more. The results, however, do seem to show some additional points of interest. While the test was timed to one half hour, it was found that less than five per cent of the pupils used more than 20 minutes.—*Taken from F. T. Rorer*, An Arithmetical Test in Three Philadelphia High Schools. The Mathematics Teacher, III, (June, 1911) *pp. 185–186.*

Letters to the editor

Dear Editor,

As usual I have read the most recent issue of THE MATHEMATICS TEACHER with considerable interest. I feel moved to comment on a statement in one of the book reviews.

In the review of *Modern Trigonometry*, a text by Brixey and Andree, the statement that the roots of a quadratic equation are "$x=4$ *and* $x=-1$" is said to appear on page 11 and is called an error.

My copy of *Modern Trigonometry* reads in part as follows: "This is true if and only if *either* $(x-4)=0$ *or* $(x+1)=0$. The *two* resulting *linear equations* may be solved to obtain $x=4$ *and* $x=-1$. *Either* of the values 4 *or* -1 will satisfy the *original equation*, . . . " (Underlining is mine). I fail to find where the authors have said that 4 *and* -1 are the roots of the quadratic equation.

I would also raise the question whether the use of "and" in giving the roots of a quadratic equation is really an "error." If Calvin Butterball finds that an equation of degree two is satisfied by either 4 or -1 he knows that each of these values is a root of the equation. Since *each* value is a root of the equation, is it not correct usage for Calvin to say that the roots are 4 *and* -1? Similarly, if Calvin finds that two equations are satisfied by $x=3$ and $y=4$ *or* by $x=-4$ and $y=-3$ he may say that solutions of the system are $(3, 4)$ *and* $(-4, -3)$?

Sincerely yours,
JOHN A. SCHUMAKER
State of New Jersey
State Teachers College
Montclair, New Jersey

Dear Editor,

I have read Professor Schumaker's comment on my review of Brixey and Andree's *Modern Trigonometry* with a great deal of interest. There is no denying that he quotes correctly. However, I suggest that he look not only at page 11 but also at subsequent pages where numbers eligible to be roots of equations are discussed. The authors are by no means consistent in their treatment of such numbers; sometimes "and" is used; sometimes "or" is used.

With respect to the question of which is correct "and" or "or", I should like to consider the solution of the following quadratic equation:

(1) $\qquad x^2-5x+6=0.$

As it stands, (1) is a statement function. The direction "solve" seems to mean "find all true substitution instances of $x^2-5x+6=0$."

The process of solving involves two things: (a) we try to find a statement function simpler than the original, such that its true substitution instances are obvious, and such that every substitution instance of the original statement function is a substitution instance of the simpler statement function; (b) we must assure ourselves that these numbers are indeed true substitution instances of $x^2-5x+6=0$.

We proceed:

$x^2-5x+6=0$	Given
x^2-5x+6 $=(x-2)(x-3)$	On the basis of the axioms of algebra
$(x-2)(x-3)=0$	By the transitive property of "$=$"
$[(x-2)(x-3)=0]\rightarrow$ $[(x-2=0)\vee(x-3=0)]$	From the theorem of algebra: $(a,\ b)\ [(ab=0)\rightarrow((a=0)\vee(b=0))]$—a theorem which is *not true* if we write "$(a=0)$ and $(b=0)$"
$(x-2=0)\vee(x-3=0)$	By modus ponens

In short, what has been shown is that

$$(x^2-5x+6=0)\rightarrow[(x-2=0)\vee(x-3=0)].$$

By use of the Generalization Principle—where we tacitly understand x to be a number—we have

$$(x)\,\{(x^2-5x+6=0)\rightarrow[(x-2=0)\vee(x-3=0)]\}.$$

A translation of the last statement is: For every x (where x is a number), if

$$x^2-5x+6=0, \text{ then } x-2=0 \text{ or } x-3=0.$$

The true substitution instances of $x-2=0$ and $x-3=0$ are, of course, obvious.

There remains the problem of satisfying ourselves that 2 and 3 *are* true substitution instances of $x^2-5x+6=0$, but this can be done in the usual fashion by direct substitution. Now, if both 2 and 3 are true substitution instances of $x^2-5x+6=0$, then we are quite correct in saying "2 and 3 are roots of $x^2-5x+6=0$."

Another way in which we might examine the question at issue is the following. Just what does it mean logically to say

$$x=2 \text{ and } x=3?$$

This statement means that x is 2 and 3 simultaneously, which is absurd.

Sincerely yours,
Myron F. Rosskopf
Professor of Mathematics

NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Minutes of the Annual Business Session

Hotel Schroeder, Milwaukee, Wisconsin

April 13, 1956

Mrs. Marie S. Wilcox, President, called the meeting to order at 4:30 P.M.

I. Miss Virginia Lee Pratt moved, seconded by Mr. Joseph Urbancek, that the minutes of the April 15, 1955 meeting be approved as printed. The motion carried.

II. Mr. M. H. Ahrendt, Executive Secretary, gave a preliminary financial report. He stated that the final report would not appear until after the close of the fiscal year which ends on June 1. The completed report will be published. Mr. Ahrendt then reported on the new office arrangements, the new membership files, and gave a brief description of his work over the past year.

III. Mr. Houston T. Karnes, Recording Secretary, gave a brief review of the actions of the Board during the past year. This review included items which were thought to be of interest to the members. The review follows:

A. All official notices must be carried in both THE MATHEMATICS TEACHER and *The Arithmetic Teacher*.
B. *The Arithmetic Teacher*, beginning with the October 1955 issue, would have six issues per year.
C. Student Membership is now available to all full-time students.
D. No meeting for Christmas 1957 is being scheduled. This is experimental in view of recommendations for only two meetings per year.
E. The Curriculum Committee at the Secondary level has been given permanent status and will be increased to a total of fifteen members. Funds are to be secured for a major study.
F. Future meetings: Summer 1956, Los Angeles, California; Christmas 1956, Jonesboro, Arkansas; Spring 1957, Philadelphia, Pennsylvania; Summer 1957, Northfield, Minnesota; Spring 1958, Cleveland, Ohio; Summer 1958, Greeley, Colorado; Spring 1959 or 1960, Buffalo, New York.
G. The report of the Membership Committee shows a fine increase in membership during the past year. Future plans of the Committee should result in an excellent continued growth.
H. A Directory of the Membership of the Council is to be published. The names will be listed both in alphabetical order and by States.
I. 1957 is the Centennial year for N.E.A. The Council is planning to cooperate in the celebration.
J. Mrs. Wilcox announced to the Board that contacts had been made with the National Science Foundation in an effort to secure funds for:
   1. A Writing Conference for the Yearbook on Concepts.
   2. The publication and distribution of a Mathematical Guidance Leaflet. Although the final decision of the Foundation has not been given, the request was favorably received.

IV. Mr. Maurice Hartung, Chairman of the Nomination and Election Committee, gave the following report of the recent balloting:

President: Mr. Howard Fehr
Vice-President (Senior High School): Mr. Donovan Johnson
Vice-President (Elementary): Miss Laura K. Eads
Directors: Mr. Phillip Jones
Mr. Philip Peak
Mr. H. Vernon Price

V. On behalf of Mr. F. L. Wren, Chairman of the new Nomination and Election Committee, who was not able to attend due to illness, Mr. Maurice Hartung explained the procedure of the Committee and invited recommendations from the membership.

VI. Mrs. Wilcox introduced the newly elected officers and directors and invited Mr. Fehr to address the meeting.

VII. Mr. Fehr expressed his appreciation for the honor which had been bestowed upon him and asked for the full support of the membership in the great task that will be his during the next two years. He then spoke of the great

progress the Council had made during the past ten years and of the Council's place and responsibility in the field of education.

VIII. There being no old business, Mrs. Wilcox called for new business.

IX. Mr. H. C. Christofferson moved, seconded by Mr. Anton S. Richert, that the retiring president and the officers and directors be given a vote of thanks for the splendid job which had been done. The motion passed.

X. Mr. Phillip Jones moved, seconded by Miss Lurnice Begnaud, that Miss Margaret Joseph and her committee members, be extended a vote of thanks for the excellent arrangements for the Milwaukee meeting. The motion passed.

XI. There being no further business, the meeting was duly adjourned at 4:55 P.M.

Respectfully submitted,
Houston T. Karnes, Recording Secretary

Your professional dates

NCTM convention dates

CHRISTMAS MEETING

December 27–29, 1956
Arkansas State College, Jonesboro, Arkansas
Lyle J. Dixon, Arkansas State College, State College, Arkansas

ANNUAL MEETING

March 28–30, 1957
Bellevue-Stratford Hotel, Philadelphia, Pennsylvania
M. Albert Linton, William Penn Charter School, Philadelphia, Pennsylvania

JOINT MEETING WITH NEA

July 1, 1957
Philadelphia, Pennsylvania
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D. C.

SUMMER MEETING

August 19–21, 1957
Carleton College, Northfield, Minnesota
Margaret Linster, St. Louis Park High School, Minneapolis 16, Minnesota, or Kenneth O. May, Carleton College, Northfield, Minnesota

Other professional dates

Fall Meeting of the Oregon Council of Teachers of Mathematics

November 3, 1956
Willamette University, Salem, Oregon
Oscar Schaaf, 123 Leigh, Eugene, Oregon

Chicago Elementary Teachers Mathematics Club

December 10, 1956
Toffenetti's Restaurant, 65 West Monroe Street, Chicago, Illinois
Genevieve E. Johnson, Volta School, Chicago, Illinois

Minutes of the Seventh Delegate Assembly

The National Council of Teachers of Mathematics
Schroeder Hotel, Milwaukee, Wisconsin
April 11–14, 1956

Edited by H. Glenn Ayre
Western Illinois State College, Macomb, Illinois

The Seventh Delegate Assembly of the National Council of Teachers of Mathematics met in the East Room of the Schroeder Hotel in Milwaukee, Wisconsin at 9:00 A.M., Thursday April 12, 1956, with Glenn Ayre presiding.

In his opening remarks, Dr. Ayre gave some history of the National Council and stated that 44 states now have a total of 64 affiliated groups. Marie S. Wilcox expressed appreciation for the support she received from the Council during the past two years. She showed how certain changes in the organization have made it more democratic. Some of these changes are: a separate editor for each publication, an increased number of committees, the Delegate Assembly, and the change in nomination procedure. This last change enables any group to send nominations to the Nomination Committee chairman, F. Lynwood Wren, George Peabody College for Teachers, Nashville, Tennessee. Delegates were introduced by their regional representatives.

Projects for affiliated groups

Donovan Johnson moderated a panel on "Projects for Affiliated Groups." Clarence Olander discussed the Minnesota meeting, stating that their meeting is held on Friday and Saturday, that the most successful sessions were the discussion sections. He explained that a 4:00 P.M. Friday meeting is popular with local teachers, and that classes were often taken by science teachers who had similar meetings on other days. Further discussion on this point showed that some local groups meet monthly, and that these monthly meetings may be on a topic chosen for the entire year, such as, "Slide Rule," or "Applications from Industry." Some groups have one or two meetings a year with outside speakers, some meet with state education associations. Michigan holds one meeting a year which includes a Friday evening, Saturday, and Sunday morning. Illinois has an annual meeting in the fall on a Saturday with six sectional meetings during the spring. New England has a one-week institute in the summer. Suggestions for projects (other than meetings) were: to be hosts to a National Council meeting, to sponsor contests, fairs, newsletters, and student written publications (separately, or as part of the newsletter).

Suggestions to gain new members were: to send requests to school administrators to allow each teacher to attend one meeting a year; to have membership blanks available at every meeting; to urge boards of education to give credit to those who attend meetings, institutes, and workshops; and to send newsletters to nonmembers.

Methods of financing programs and projects were discussed. In general, membership dues are one dollar, although some are less. Some funds have been obtained through local industries, county superintendents, and state education associations. Colleges, universities, and boards of education sometimes share expenses when meetings are held on the campus. It was suggested that foundations might provide funds.

Contests

Daniel Lloyd reported on the work of the Contest Committee and stated that the committee's report would be published in an early issue of THE MATHEMATICS TEACHER.

Number and time of council meetings

Mary Potter served as moderator for this discussion. She gave a brief history of Council meetings and presented this information on attendance for the last few years:

	High	*Low*	*Average*
Annual	1322	619	956
Christmas	851	223	425
Summer	518	290	405
NEA (joint meeting)	174	62	121

Myrl Ahrendt reported that the trend is toward an increasing attendance at all meetings. He further stated that about 50 per cent of those who attend all meetings travel less than 200 miles to do so. John Brown mentioned that the Mathematics Association of America holds two national meetings a year, January and early fall.

Those who opposed four meetings a year felt they were too costly, and that our leaders were "pushed a bit" to plan and attend them all. An argument in favor was that four meetings a year provided more convenient times for more members to attend a meeting, and of course, the meetings are a stimulus to local and regional professional work.

An argument used against the Christmas meeting was that attendance at this meeting prevented the members from spending the vacation with their families.

Some favored a regional meeting instead of one or two of the national meetings because of the convenience of attending and the opportunity to make better use of local talent. Others opposed because they feared such a meeting would lack the prestige of a national meeting and would eventually lose appeal.

The following expression of opinion was taken to advise the Board of Directors in the matter of number and time of meetings:

1. One annual meeting in the spring plus six regional	12 in favor
2. One annual meeting in the spring plus Christmas	8 in favor
3. One annual meeting in the spring plus summer	16 in favor
4. Two meetings (Board to determine time)	32 in favor
5. No change from present plan	8 in favor

Pamphlet on clubs

Henry Swain asked for suggestions for activities and projects which would be suitable for inclusion in a proposed pamphlet for high school and junior high school mathematics clubs.

State representatives

Questions on this matter were deferred until a later meeting since there was such a short time in which to discuss it.

Continuation of newsletter

Sixteen favored continuing the newsletter—none were opposed. The meeting was then adjourned.

Respectfully submitted,
Nellie Alexander,
Joe Kennedy,
Secretaries of Delegate Assembly

The following people were chosen by their local groups as official delegates and alternates to the Delegate Assembly. The name listed first is that of the delegate. If two names are listed, the second one is that of the alternate. Although some were unable to be present, the attendance was good and the affiliated groups were well represented.

Department of Mathematics, Alabama Education Association
Dr. R. E. Wheeler, Howard College, Birmingham
Mrs. Paul Jones, Shades Valley High School, Birmingham

Arkansas Council of Teachers of Mathematics
Charles G. Pitner, Harding College, Searey
Christine Poindexter, 4219 West 13th Street, Little Rock

California Mathematics Council
James Nudelman, Mountain View High School, Mountain View
L. Clark Lay, 593 West Montana Street, Pasadena City College

Colorado Council of Teachers of Mathematics
Ida Kammerzell, Route 1, Box 30, Johnstown

The District of Columbia Teachers of Mathematics
Mrs. Faith Novinger, 1227 Longfellow Street, N. W.
Jane M. Hill, 3051 Harrison Street, N. W.

Benjamin Banneker Mathematics Club
Mrs. Juanita Tolson, 723 Jefferson Street, Washington, D. C.
Mrs. Jonelle B. Washington, 116 Longfellow Street, N. W.

Florida Council of Teachers of Mathematics
Dr. Robert S. Fouch, 227 Westridge Drive, Tallahassee
Dr. William A. Gager, 2960 Parkway, Gainesville

Dade County Council of Teachers of Mathematics
Mr. Agnes Rickey, Miami Edison Senior High School, 6101 N. W. Second Street, Miami

Pinellas County Council of Teachers of Mathematics
John Tilley, 4480 40th Street North, St. Petersburg
William C. Smith, 634 Belleview, Clearwater

Georgia Mathematics Council
Sara Durham, Coffee County High School, Douglas

Illinois Council of Teachers of Mathematics
Clarence Fuqua, High School, Champaign
Leroy Sachs, Kahokia Commonfields High School, East St. Louis, Illinois

Chicago Elementary Teacher's Mathematics Club
J. J. Urbancek, Chicago Teachers College, 6800 Stewart Avenue, Chicago 21
Ramona Goldblatt, Burley School, 1630 West Barry Avenue, Chicago 13

Men's Mathematics Club of Chicago and Metropolitan Area
Hobert Sistler, J. Sterling Morton High School and Junior College, Cicero
H. Steubner, 339 Western Avenue, Clarendon Hills

Women's Mathematics Club of Chicago and Vicinity
Jean Bryson, 425 Surf Street, Chicago 14
Edith Benjamin, Warren Township High School, Gurnee

Indiana Council of Teachers of Mathematics
Grace Arbogast, 1829 East Main Street, Richmond
Philip Peak, School of Education, Indiana University, Bloomington

Gary Council of Teachers of Mathematics
George Orlich, R. R. 1, County Line Road, Hobart

Iowa Association of Mathematics Teachers
Dale Jungst, Carroll Public Schools, Carroll
Margaret Molison, Williamsburg

Wichita Mathematics Teachers Association
Anton S. Richert, East High School, Wichita
Arno E. Manke, 1227 Harding, Hutchinson

Mathematics Section of the Maryland State Teachers Association
Helen Cooper, Bethesda High School, Bethesda
Mathematics Section of Prince George's County Teachers Association
Louis Halter, Greenbelt High School, Greenbelt
Michigan Council of Teachers of Mathematics
Nickoline Bye, 1109 South Washington, Mt. Pleasant
Murel Kilpatrick, 114 North Hamilton, Ypsilanti
Detroit Mathematics Club
Lucille Martin, 5074 Fernwood Avenue, Detroit 4
Minnesota Council of Teachers of Mathematics
Clarence Olander, St. Louis Park High School, St. Louis, Minnesota
Donovan Johnson, University High School, University of Minnesota, Minneapolis
Missouri Affiliated Group of the National Council of Teachers of Mathematics
Frances Story, St. Charles, Missouri
Nebraska Section—National Council of Teachers of Mathematics
Virginia Lee Pratt, 5123 Underwood Avenue, Omaha 3
John Bryan, 2922 Izard Street, Omaha
The Association of Teachers of Mathematics in New England
Henry W. Syer, 332 Bay State Road, Boston, Massachusetts
Jackson B. Adkins, Box 49, Exeter, New Hampshire
Southern New England Preparatory School Mathematics Association
Robert Rourke, Kent, Connecticut
Association of Mathematics Teachers of New Jersey
Max A. Sobel, Robert Treat High School, Newark
John K. Reckzeh, Jersey City State Teachers College, Jersey City
Mathematics Section of the New Mexico Education Association
Reba Jinkins, Clouis High School, Clouis
Association of Mathematics Teachers of New York State
Lucille E. Brooks, Marcellus Central School, Marcellus
Florence Deci, Palmyra Macedon Central School, Palymra
Association of Teachers of Mathematics of New York City
Leona Freeman, Morris High School, 166th Street and Morris Avenue, Bronx, N. Y.
Nassau County Mathematics Teachers Association
Emily J. Klein, Bethpage High School, Bethpage, N. Y.
Department of Mathematics of the North Carolina Education Association
Annie John Williams, Durham
Ohio Council of Teachers of Mathematics
Clarence Heinke, Capitol University, Columbus 9
Charles E. Scott, 858 Roanoke Road, Cleveland Heights 21
Mathematics Club of Greater Cincinnati
Dr. Robert Price, 1131 Madeleine Circle, Cincinnati 31
Eleanor J. Graham, 3529 Shaw Avenue, Cincinnati 8
Cleveland Mathematics Club
Alex Rubins, 2529 Edgehill Road, Cleveland Heights 6
Jim Gates, Maple Heights High School, Maple Heights
Greater Toledo Council of Teachers of Mathematics
Dr. Richard Shoemaker, 2647 Thoman Place, Toledo 13
Violet Davis, 1914 Evansdale, Toledo 7
Oklahoma Council of Teachers of Mathematics
Florence Ingham, 1311 Delaware Avenue, Bartlesville
Eunice Lewis, University School, Norman
Tulsa Chapter of National Council of Teachers of Mathematics
Mrs. Carolyn Stewart, 4920 East 25th Street, Tulsa
Mr. Coy C. Pruitt, 1727 South College, Tulsa
Ontario Association of Teachers of Mathematics and Physics
Harold W. Harris, 124 Ranlugh Avenue, Toronto
Oregon Council of Teachers of Mathematics
Lesta Hoel, Public Schools, Portland
Pennsylvania Council of Teachers of Mathematics
Lee E. Boyer, State Teachers College, Millersville
Catherine A. V. Lyons, 12 South Fremont Avenue, Pittsburgh 2
Western Pennyslvania Association of Teachers of Mathematics
Melvin Veseley, Carrick High School, 125 Parkfield Street, Pittsburgh 10
Dr. John C. Knipp, 323 Morewood Avenue, Pittsburgh 6
Association of Teachers of Mathematics of Philadelphia and Vicinity
Karl S. Kalman, Lincoln High School, Ryan and Rowland Streets, Philadelphia 36
Texas Council of Teachers of Mathematics
Mrs. Lorena Holder, 312 West Ninth Street, Dallas 8
Houston Council of Teachers of Mathematics
Margaret Kennerly, 3214 Huntingdon, Houston
Fern Saxton, 205 Drouet, Houston
Mathematics Section, Virginia Education Association
Mrs. Howard McCue, St. Catherine's School, Richmond
Emilie Holladay, Newport News High School, Newport News
Richmond Branch of National Council of Teachers of Mathematics
Lucy Waring, Highland Springs High School, Highland Springs

The Puget Sound Council of Teachers of Mathematics
Elizabeth Roudebush, 815 Fourth Avenue North, Seattle
West Virginia Council of Mathematics Teachers
Dorothy Bragonier, Marshall College, Huntington
Mrs. Robert Thomson, Woodrow Wilson, Beckley
Wisconsin Mathematics Council
J. A. Brown, Wisconsin High School, Madison
Joseph Kennedy, Wisconsin High School, Madison

Can two lines coincide?

Near the beginning of the year in plane geometry, I had led the class to investigate possible relations that one straight line might have with another in space. The purpose in this was to lead up to the need for careful definitions of the terms *intersecting, perpendicular, parallel,* and *skew lines.* The plan succeeded and, for high school students, the members of the class made fairly acceptable definitions, later refined and officially accepted in group discussion.

Several students went further, wondering whether two lines, lying one on top of the other, made a relation worth mentioning. Most of the class felt that it was not, but that it was an interesting academic question. I told them that this notion actually serves a purpose in analytic geometry (where $2x-3y=10$ and $4x-6y=20$ are called equations of *coincident* lines). On the strength of this information, the class was willing to add this definition: *Coinciding lines are two or more lines that lie on top of each other.* Although we discussed that this meant the lines occupied the same space, we did not refine the definition.

It was several weeks later that one thoughtful lad in the class saw a contradiction. He had been thinking about the definition above and about one of our basic assumptions: *One and only one straight line can be drawn through two points.* He raised the following questions, and stated them fairly well, too:

1. If we can say that two lines coincide, how can we also say that through two points *only one* straight line can be drawn?
2. If only one line can be drawn through two points, how can there *be* such a thing as coinciding lines?

Certainly this incident shows an occasion where a student did some critical mathematical thinking. Obviously, it is a situation pointing up the need for careful definitions. I think it has even deeper significance. Without putting it into so many words, this high school lad told me, "If you believe in Euclid's foundations of geometry as a model for space, you can't also use the analytic geometry model (and still have coinciding lines)" I believe I agree with him. Do you?—*Lyman C. Peck, Ohio Wesleyan*

Because a situation is socially significant, it does not necessarily follow that it is also mathematically meaningful.—*This statement is based on the study by Dan T. Dawson.* Number Grouping as a Function of Complexity, *Ed.D., 1952, Stanford University, Palo Alto, California: Major Faculty Adviser, Dr. Paul R. Hanna.*

• NOTES FROM THE WASHINGTON OFFICE

by M. H. Ahrendt, Executive Secretary, NCTM, Washington, D. C.

Below are the official reports of registrations for four recent meetings of the Council. These reports are not attendance reports in the true sense. They include only persons who registered for the purpose of participating in the meeting. Thus, members of families and casual visitors are not included. Also not included, of course, are persons who may have participated in the convention without registering. It is sincerely hoped that there are not many persons in this latter category.

One highlight of the conventions, for which reports are given here, was the large attendance at the Sixteenth Christmas Meeting. This meeting was the largest Christmas meeting in the history of the Council, and was larger than many previous annual meetings.

Registrations at the Sixteenth Christmas Meeting

The National Council of Teachers of Mathematics, Washington, D. C., December 27–29, 1955

Alabama	1	Nebraska	8
Arkansas	2	New Hampshire	4
California	1	New Jersey	39
Colorado	1	New Mexico	1
Connecticut	9	New York	52
District of Columbia	288	North Carolina	14
Florida	6	Ohio	30
Georgia	1	Oklahoma	1
Illinois	27	Pennsylvania	44
Indiana	5	Rhode Island	3
Iowa	4	South Carolina	3
Kentucky	1	Tennessee	4
Louisiana	2	Texas	2
Maine	1	Virginia	89
Maryland	160	West Virginia	5
Massachusetts	15	Wisconsin	2
Michigan	14	Canada	5
Minnesota	2	Foreign	1
Missouri	3		
Montana	1	Total	851

Registrations at Annual Joint Meeting with the NEA

The National Council of Teachers of Mathematics, Chicago, Illinois, July 4, 1955

Alabama	1	Ohio	2
California	4	Pennsylvania	3
Connecticut	1	South Dakota	2
Delaware	1	Tennessee	1
District of Columbia	1	Texas	1
Georgia	1	Virginia	1
Illinois	25	Washington	1
Indiana	3	West Virginia	1
Iowa	1	Wisconsin	4
Michigan	3	Foreign	2
Nebraska	1		
New Jersey	2	Total	62

Registrations at the Fifteenth Summer Meeting

The National Council of Teachers of Mathematics, Bloomington, Indiana, August 21–24, 1955

Arkansas	1	Missouri	3
California	8	Nebraska	6
Colorado	2	New Jersey	1
District of Columbia	3	New Mexico	2
Florida	4	New York	2
Georgia	2	North Carolina	1
Illinois	76	Ohio	32
Indiana	145	Oklahoma	1
Iowa	14	Oregon	1
Kansas	4	Pennsylvania	13
Kentucky	6	Tennessee	7
Louisiana	4	Texas	8
Maryland	1	Virginia	3
Massachusetts	1	Wisconsin	12
Michigan	16		
Minnesota	5	Total	384

Registrations at the Thirty-fourth Annual Meeting

The National Council of Teachers of Mathematics, Milwaukee, Wisconsin, April 11–14, 1956

Alabama	2	New Mexico	2
Arkansas	1	New York	26
California	6	North Carolina	1
Connecticut	2	North Dakota	2
District of Columbia	12	Ohio	35
Florida	6	Oklahoma	6
Georgia	6	Pennsylvania	10
Illinois	262	Rhode Island	1
Indiana	37	South Carolina	1
Iowa	34	South Dakota	1
Kansas	3	Tennessee	5
Louisiana	4	Texas	7
Maryland	5	Virginia	5
Massachusetts	9	Washington	2
Michigan	37	West Virginia	1
Minnesota	26	Wisconsin	559
Mississippi	2	Wyoming	1
Missouri	6	Canada	8
Nebraska	6		
New Hampshire	2	TOTAL	1,151
New Jersey	10		

The most marked shortcomings of the mathematics curriculum are in the areas of the collection and interpretation of data.—*This statement is based on the study by Max Beberman.* The Teaching of Statistics in Secondary School Mathematics, *Ed.D., 1952, Teachers College, Columbia University, New York City; Major Faculty Adviser, Professor Howard F. Fehr.*

Have you read?

KLINE, MORRIS, "The Straight Line," *Scientific American*, March 1956, pp. 105–114.

Most geometry teachers do not define a straight line and therefore by default the geometry student goes through mathematics with only the physical concept of it.

Here is a well-written article which brings to the fore the depth of logic in the simple structure of the straight line.

Most of you realize the defects of Euclid's definition and you are familiar with the attempts of Gauss, Lobachevski, Bolyai and Reimann to remove these defects. But have you ever stopped to think how sets of points of zero dimensions can make a line of one dimension, or why a surface is two dimensional? Have you ever wondered what happens to the shortest distance between two points as the surfaces containing these points vary or why do all lines contain the same number of points regardless of the length of each, and how a finite length may appear infinite?

Most of all you will want to read this and have your students read it to see how the single, simple straight line leads to modern mathematics which has proved to be the foundation and strength of modern science.

SOLOW, HERBERT, "Operations Research Is in Business," *Fortune*, February, 1956, pp. 128–131.

A new profession for your mathematics students is "Operations Research." These are groups of scientists and mathematicians who have organized consultant services for the businessman. To define their activity we might say it is the scientific study of complex organizations aimed at identifying problems and giving executives a basis for making quantitative decisions.

This article gives one case and shows how they operate. It is "The Walking Drag Line Case." It will surprise you to see the mathematics involved. Angles of rotation, angles of elevation, width of cut, position of dump, length of walk and many others. The variables and parameters are analyzed.

In this problem there were ten parameters and a set of equations were developed and the mathematical model was solved by high speed computing machines for varying variables. These solutions permitted the selection of an aptemin combination of variables. Have your interested students read this article for information about mathematics and a growing profession.—PHILIP PEAK

STRAIGHT LINE UNIT, NO. 186

This interesting kit discusses the problem of drawing an original straight line without the use of a straightedge.

Shows how the straight line problem can be solved by the use of linkages. Also explains other uses of linkages.

Contains equipment for constructing several small operating models of linkages. Instruction leaflet gives directions, with bibliography for further reading.

75¢ each. 3 for $1.50.

National Council of Teachers of Mathematics
1201 Sixteenth Street, N. W.
Washington 6, D. C.

BOOKS ESSENTIAL FOR REFERENCE AND CLASS

DR. BRUHN'S TABLES OF LOGARITHMS; 7 Place Logarithmic & Trigonometrical Tables. For solving the most complicated and extensive numerical calculations by simple addition or subtraction. Postpaid $4.00

ADDITION & SUBTRACTION LOGARITHMS (Gaussian Tables) to 7 Decimal Places. For solving formulas involving Addition and Subtraction without adapting them to logarithmic computation. Postpaid $2.00

BENSON'S NATURAL TRIGONOMETRICAL FUNCTIONS. Contains the Natural Sine, Cosine, Tangent and Cotangent to 7 Decimal Places for every 10 Seconds of Arc from 0 Degrees, Semi-Quadrantly arranged. Postpaid $5.00

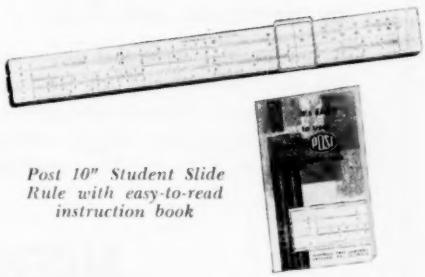
THE CHARLES T. POWNER CO.
Dept. TMT, 407 S. Dearborn St., Chicago 5, Ill.

Why does Johnny get wrong answers on humid days?

IT isn't that Johnny is mentally slower than the others. It is simply that his "bargain" slide rule isn't much of a bargain after all. His slide rule swells or contracts with changes in heat and humidity. Warpage occurs and the readings are undependable. Then, to make things still worse, the slide binds and becomes difficult to adjust. With these disadvantages, Johnny starts a possibly promising career with a severe handicap!

Was this a wise investment?

For only $3.75 (special classroom price—$2.81) he could have purchased a Post 10" Student Slide Rule (Mannheim type). This seasoned, laminated bamboo slide rule does not warp or shrink; does not bind, stick or require artificial lubricants; and is not affected by climatic conditions. Accurate graduations are *machine cut* into the white celluloid face, making them a permanent part of the rule. Even the cursor is designed for durability and dependability . . . it is framed in metal and has a tension spring maintaining the vertical hairline.



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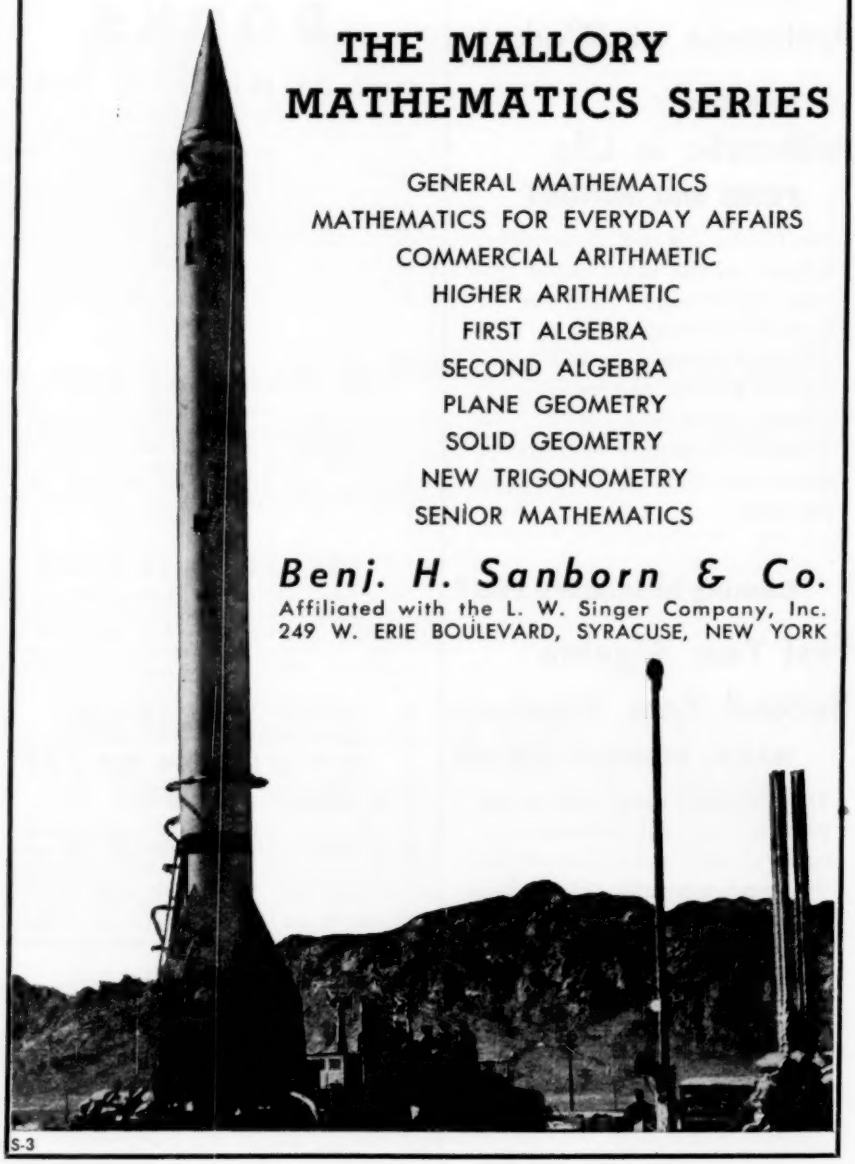

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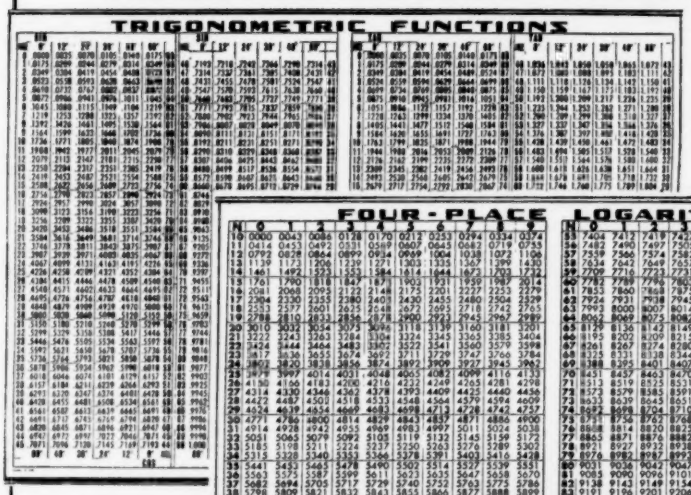

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